Finite dimensional Hilbert space and frame quantization

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Abstract. The quantum observables used in the case of quantum systems with finite-dimensional Hilbert space are defined either algebraically in terms of an orthonormal basis and discrete Fourier transformation or by using a continuous system of coherent states. We present an alternative approach to these important quantum systems based on the finite frame quantization. Finite systems of coherent states, usually called finite tight frames, can be defined in a natural way in the case of finite quantum systems. Novel examples of such tight frames are presented. The quantum observables used in our approach are obtained by starting from certain classical observables described by functions defined on the discrete phase space corresponding to the system. They are obtained by using a finite frame and a Klauder-Berezin-Toeplitz type quantization. Semi-classical aspects of tight frames are studied through lower symbols of basic classical observables.

1. Introduction

Weyl's formulation of quantum mechanics [46] opened the possibility of studying the dynamics of quantum systems both in infinite-dimensional and finite-dimensional systems. Based on Weyl's approach, generalized by Schwinger [38], several authors (e.g., [35, 36, 39, 40, 10, 22, 47, 28, 26, 41, 42, 43, 44, 37]) investigated the analogue of the harmonic oscillator in finite-dimensional Hilbert spaces. This work combines concepts from quantum mechanics with discrete mathematics (number theory, finite sums [8], combinatorics, etc).

The general formalism of coherent states [33, 34, 17, 48, 15] can also be used in the context of finite-dimensional quantum systems. This powerful method defines continuous systems of coherent states as well as finite systems of coherent states, usually called *finite tight frames* [3, 11, 12, 18]. In finite quantum systems mainly continuous systems of coherent states have been studied. Among recent explorations of finite systems of coherent states, see for instance [13, 15]. In this paper we study finite frames in the context of finite quantum systems. This is a *finite frame quantization*, and it is a finite version of a conventional Klauder-Berezin-Toeplitz type coherent state quantization.

In section 2 we briefly present topics from the theory of finite quantum systems, which are needed later. In section 3 we present the general formalism for finite-dimensional tight frames and the related quantization. We define covariant and contravariant symbols and we briefly recall and complete interesting probabilistic and semiclassical aspects of the coherent states/frame formalism which have been developed in previous works [2, 12]. We also define Wigner functions and Weyl functions in this context. In section 4 we present various examples of tight frames. The first example is a 'lattice frame' and consists of d^2 vectors (where d is the dimension of the Hilbert space). It depends on a 'fiducial' vector, and we use an eigenvector of the discrete Fourier transformation which is related to the harmonic oscillator vacuum through a Zak [50] or Weil [45] transform. This example has been previously defined in [49] in terms of the Perelomov method by starting from a discrete version of the Heisenberg group. The second example is a 'sparse frame' and consists of d^2/n vectors where n is a divisor of d such that the $(n, d/n)$ are coprime. This is a novel example and is discussed in section 4.2.

In section 5 we show how the formalism can be used for practical calculations (matrix elements, spectra, etc) in the quantization context. We conclude in section 6 with a discussion of our results.

2. Finite quantum systems

We consider a quantum system where position and momentum take values in the ring $\mathbb{Z}_d = \mathbb{Z}/d\mathbb{Z} = \{0, 1, ..., d-1\}$ of the integers modulo d, where d is a fixed positive integer. The Hilbert space $\mathcal{H}$ of this system is d-dimensional, and we describe it by using the orthonormal basis of 'position states' $\{|e_0\rangle, |e_1\rangle, ..., |e_{d-1}\rangle\}$. The finite Fourier transform

$$F:\mathcal{H}\longrightarrow\mathcal{H},\qquad F=\frac{1}{\sqrt{d}}\sum_{k,l=0}^{d-1}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kl}|e_k\rangle\langle e_l| \tag{1}$$

allows us to consider the directly related orthonormal basis of 'momentum states' $\{|f_0\rangle, |f_1\rangle, ..., |f_{d-1}\rangle\}$ satisfying the relations

$$|f_k\rangle=F|e_k\rangle=\frac{1}{\sqrt{d}}\sum_{l=0}^{d-1}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kl}|e_l\rangle\,,\qquad |e_l\rangle=F^+|f_l\rangle=\frac{1}{\sqrt{d}}\sum_{k=0}^{d-1}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}lk}|f_k\rangle\,.$$

Each state $|\psi\rangle\in\mathcal{H}$ can be expanded as

$$|\psi\rangle=\sum_{l=0}^{d-1}\psi_l|e_l\rangle=\sum_{k=0}^{d-1}\tilde{\psi}_k|f_k\rangle$$

where the functions $\psi:\mathbb{Z}_d\longrightarrow\mathbb{C}: l\mapsto\psi_l$ and $\tilde{\psi}:\mathbb{Z}_d\longrightarrow\mathbb{C}: k\mapsto\tilde{\psi}_k$ satisfying

$$\psi_l=\frac{1}{\sqrt{d}}\sum_{k=0}^{d-1}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}lk}\tilde{\psi}_k\,,\qquad \tilde{\psi}_k=\frac{1}{\sqrt{d}}\sum_{l=0}^{d-1}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}kl}\psi_l\,. \tag{2}$$

are the corresponding 'wavefunctions' in the position and momentum representations.

The parity operator with respect to the origin is defined as $P(0,0)=F^2$ and $P(0,0)^2=F^4=\mathbb{I}_{\mathcal{H}}$. The self-adjoint 'position' and 'momentum' operators $x,p:\mathcal{H}\longrightarrow\mathcal{H}$,

$$x=\sum_{l=0}^{d-1}l\,|e_l\rangle\langle e_l|\,,\qquad p=\sum_{k=0}^{d-1}k\,|f_k\rangle\langle f_k|\,, \tag{3}$$

are defined modulo d and satisfy the relations

$$\begin{array}{lll} FxF^+=p\,, & x|e_l\rangle=l|e_l\rangle\,, & x|f_k\rangle=\frac{1}{d}\sum_{l,j=0}^{d-1}l\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}l(k-j)}|f_j\rangle\,,\\ FpF^+=-x\,, & p|f_k\rangle=k|f_k\rangle\,, & p|e_l\rangle=\frac{1}{d}\sum_{k,m=0}^{d-1}k\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k(m-l)}|e_m\rangle\,. \end{array}$$

The function [42]

$$\Delta_0(x)=\frac{1}{d}\sum_{l=0}^{d-1}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}lx}\,, \tag{4}$$

and its derivatives

$$\Delta_s(x)=\frac{\mathrm{d}^s}{\mathrm{d}x^s}\Delta_0(x)=\frac{1}{d}\sum_{l=0}^{d-1}\left(\frac{2\pi\mathrm{i}}{d}l\right)^s\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}lx} \tag{5}$$

are useful in the calculation of matrix elements. For $x\in\mathbb{Z}_d$, the $\Delta_0(x)=\delta(x,0)$ where $\delta(x,0)$ is the Kronecker delta in $\mathbb{Z}_d$. We can now calculate the commutator

$$\langle e_n|[x,p]|e_m\rangle=\frac{d}{2\pi i}(n-m)\Delta_1(n-m) \tag{6}$$

We note here that matrix elements of 'angular operators' like x,p involve the summation over a 'period' from N to $N+d-1$, and the result does depend on N. Only exponentials of these operators (like in the displacement operators below) are single-valued.

Table 1. List of the eigenvalues ξ_n of the commutator $[x, p]$ in the case $d = 21$.

n	ξ_n	n	ξ_n	n	ξ_n
1	-133.9652067678112 i	8	3.342253797136738 i	15	3.342253907182515 i
2	-27.11600086881775 i	9	3.342253804904009 i	16	3.342264884479544 i
3	0.5632328492846453 i	10	3.342253804929771 i	17	3.342954064008703 i
4	3.198831527436455 i	11	3.342253804929802 i	18	3.369561581989102 i
5	3.337619084687670 i	12	3.342253804929803 i	19	4.015171698810642 i
6	3.342159991389251 i	13	3.342253804930850 i	20	13.73901531616308 i
7	3.342252660619915 i	14	3.342253805426666 i	21	92.75011344338981 i

In table 1 we give the eigenvalues of this $d \times d$ matrix for the case $d = 21$. It is seen that the commutator is approximately a multiple of the unit matrix, as it should (for large d).

The displacement operators [38, 39, 49]

$$A\,, B : \mathcal{H} \longrightarrow \mathcal{H}\,, \qquad A = \mathrm{e}^{\frac{2\pi\mathrm{i}}{d}p}\,, \qquad B = \mathrm{e}^{\frac{2\pi\mathrm{i}}{d}x}\,, \tag{7}$$

are single-valued and

$$\begin{aligned} &A|e_l\rangle = |e_{l-1}\rangle\,, && A|f_k\rangle = \mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k}|f_k\rangle\,, && A^d = B^d = \mathbb{I}_{\mathcal{H}}\,, \\ &B|e_l\rangle = \mathrm{e}^{\frac{2\pi\mathrm{i}}{d}l}|e_l\rangle\,, && B|f_k\rangle = |f_{k+1}\rangle\,, && A^\alpha B^\beta = \mathrm{e}^{\frac{2\pi\mathrm{i}}{d}\alpha\beta}B^\beta A^\alpha\,. \end{aligned}$$

General displacements operators with respect to $(\alpha, \beta) \in \mathbb{Z}_d \times \mathbb{Z}_d$ are given by

$$D(\alpha, \beta) = A^\alpha B^\beta \mathrm{e}^{-\frac{\pi\mathrm{i}}{d}\alpha\beta}\,; \qquad [D(\alpha, \beta)]^\dagger = D(-\alpha, -\beta)\,. \tag{8}$$

For an arbitrary operator Θ, it has been shown [44] that

$$\frac{1}{d}\sum_{\alpha,\beta=0}^{d-1} D(\alpha, \beta)\Theta[D(\alpha, \beta)]^\dagger = \mathbb{I}_{\mathcal{H}}\mathrm{Tr}\Theta\,. \tag{9}$$

The parity operator with respect to $(\alpha, \beta) \in \mathbb{Z}_d \times \mathbb{Z}_d$ is given by

$$P(\alpha, \beta) = D(\alpha, \beta)P(0, 0)[D(\alpha, \beta)]^\dagger\,, \qquad [P(\alpha, \beta)]^2 = \mathbb{I}_{\mathcal{H}} \tag{10}$$

and it is related to the displacement operators through the Fourier transform

$$D(\alpha, \beta) = \frac{1}{d}\sum_{\gamma,\delta=0}^{d-1} \mathrm{e}^{\frac{2\pi\mathrm{i}}{d}(\alpha\delta-\beta\gamma)}P(\gamma, \delta)\,. \tag{11}$$

3. Tight finite frames and finite frame quantization

Let $\mathcal{H}$ be a d-dimensional Hilbert space and $\{|e_0\rangle, |e_1\rangle, ..., |e_{d-1}\rangle\}$ an orthonormal basis of $\mathcal{H}$. Let $\mathcal{X} = \{a_0, a_1, ..., a_{M-1}\}$ be a set of 'parameters' or 'indices', with $M \geq d$. An example will be given later with $\mathcal{X} = \mathbb{Z}_d \times \mathbb{Z}_d$ for which $M = d^2$. We consider an orthonormal system of d functions $\{\phi_k : \mathcal{X} \longrightarrow \mathbb{C}\}_{k=0}^{d-1}$:

$$\sum_{n=0}^{M-1} \overline{\phi_j(a_n)}\,\phi_k(a_n) = \delta_{jk} \tag{12}$$

such that

$$\kappa_n = \sum_{k=0}^{d-1} |\phi_k(a_n)|^2 \neq 0 \qquad \text{for any } n \in \{0, 1, ..., M-1\}\,. \tag{13}$$

It follows that

$$\sum_{n=0}^{M-1} \kappa_n = d \quad \text{and} \quad \kappa_n \leq 1\,.$$

The latter inequality is strict if $d < M$.

We then consider the map $\mathcal{X} \ni a_n \mapsto |a_n\rangle \in \mathcal{H}$ defined by

$$|a_n\rangle = \frac{1}{\sqrt{\kappa_n}} \sum_{k=0}^{d-1} \overline{\phi_k(a_n)}\, |e_k\rangle\,, \qquad n = 0, ..., M-1\,. \tag{14}$$

The M vectors $|a_n\rangle$ overlap as:

$$\langle a_n | a_m \rangle = \frac{1}{\sqrt{\kappa_n\, \kappa_m}} \sum_{k=0}^{d-1} \phi_k(a_n)\, \overline{\phi_k(a_m)}\,. \tag{15}$$

They solve the unity in $\mathcal{H}$ in the following way:

$$\sum_{n=0}^{M-1} \kappa_n\, |a_n\rangle\langle a_n| = \mathbb{I}_{\mathcal{H}}\,. \tag{16}$$

Therefore the vectors $|a_n\rangle$ form a tight finite frame in $\mathcal{H}$ [11, 12].

For later use we give the relation

$$D(\alpha, \beta)|a_n\rangle = \frac{1}{\sqrt{\kappa_n}} e^{-\frac{\pi i}{d}\alpha\beta} \sum_{k=0}^{d-1} \overline{\phi_k(a_n)}\, e^{\frac{2\pi i}{d}\beta k} |e_{k-\alpha}\rangle. \tag{17}$$

The considered tight frame $\{|a_n\rangle\}_{n=0}^{M-1}$ defines two probability distributions which can be interpreted in terms of a Bayesian duality. The latter underlies practically all overcomplete families of states in finite- or infinite- dimensional resolving the unity. The intimate connection between certain of these families and statistical distributions has been explained in [19, 20, 21, 2]:

(i) A *prior* distribution on the set of indices $k \in \{0, ..., d-1\}$, with parameter $a_n \in \mathcal{X}$,

$$k \mapsto \frac{|\phi_k(a_n)|^2}{\kappa_n} = |\langle e_k | a_n \rangle|^2\,, \tag{18}$$

with

$$\sum_{k=0}^{d-1} |\langle e_k | a_n \rangle|^2 = 1 \quad \text{for} \quad n \in \{0, 1, ..., M-1\}\,.$$

Adopting a quantum mechanical language, this probability could be considered as concerning experiments performed on the system within some experimental protocol in order to measure the spectral values of a certain self-adjoint operator (a "quantum observable") A acting in $\mathcal{H}$ (e.g. the position x in (3) and having the spectral resolution $A = \sum_k \lambda_k |e_k\rangle\langle e_k|$ (e.g. the measured positions "k" in (3). Precisely, $|\langle e_k|a_n\rangle|^2$ is the probability to get the value λ_k from a measurement of A which is performed on the system when the latter is prepared in the "state" $|a_n\rangle$.

(ii) A *posterior* distribution on the original set of parameters $a_n \in \mathcal{X}$, equipped with uniform (discrete) measure, and with parameter $k \in \{0, ..., d-1\}$,

$$a_n \mapsto |\phi_k(a_n)|^2 \,, \tag{19}$$

with

$$\sum_{n=0}^{M-1} |\sqrt{\kappa_n}|\phi_k(a_n)|^2 = 1 \,.$$

The Bayesian duality stems in the two interpretations: the resolution of the unity verified by the states $|a_n\rangle$, introduces a preferred *prior measure* on $\mathcal{X}$, which is the set of parameters of the distribution $k \mapsto \dfrac{|\phi_k(a_n)|^2}{\kappa_n}$, with this distribution itself playing the role of the *likelihood function*. The associated distributions $a_n \mapsto |\phi_k(a_n)|^2$ on the original set $\mathcal{X}$, indexed by k, become the related *conditional posterior distributions*. For a concrete example of such a duality in the discrete-continuous case, see Section 2 in [2].

3.1. Covariant and contravariant symbols

To each function $f : \mathcal{X} \longrightarrow \mathbb{C}$ we associate the operator

$$A_f : \mathcal{H} \longrightarrow \mathcal{H}, \qquad A_f = \sum_{n=0}^{M-1} \kappa_n \, f(a_n) \, |a_n\rangle\langle a_n| \,. \tag{20}$$

The matrix elements of A_f with respect to the orthonormal basis $\{|e_k\rangle\}$ are

$$\langle e_k|A_f|e_j\rangle = \sum_{n=0}^{M-1} \kappa_n \, f(a_n) \, \langle e_k|a_n\rangle\langle a_n|e_j\rangle = \sum_{n=0}^{M-1} f(a_n) \, \overline{\phi_k(a_n)} \, \phi_j(a_n) \,. \tag{21}$$

The operator A_f corresponding to a real function f is self-adjoint, and this can be regarded as a Klauder-Berezin-Toeplitz type quantization [4, 5, 6, 7, 23, 24, 25] of f, the notion of quantization being considered here in a wide sense [15]. The eigenvalues of the matrix A_f form the "quantum spectrum" of f (by opposition to its "classical spectrum" that is the set of its values $f(a_n)$).

Given an operator $A = A_f$, the function f, possibly not unique, is called *upper [29] or contravariant symbol [6] its analog in Quantum Optics is the so-called P-function*) of A. We also introduce the function $\check{f} = \check{A}_f : \ \mathcal{X} \longrightarrow \mathbb{R}$, such that

$$\begin{aligned} \check{f}(a_n) &= \langle a_n|A_f|a_n\rangle = \sum_{m=0}^{M-1} \kappa_m \, f(a_m) \, |\langle a_n|a_m\rangle|^2 \\ &= \frac{1}{\kappa_n} \sum_{m=0}^{M-1} f(a_m) \left| \sum_{k=0}^{d-1} \phi_k(a_n) \, \overline{\phi_k(a_m)} \right|^2 , \end{aligned} \tag{22}$$

which is called *lower [29] or covariant symbol [6] its analog in Quantum Optics is the so-called Q-function*) of A_f. It can be viewed as finite version of the so-called Berezin transform of the original function f (see for instance [14]).

3.2. Stochastic aspect of a finite tight frame

In [12] two of us have presented some stochastic properties displayed by finite tight frames. We give here a more complete account of these remarkable features. Let us introduce the real $M \times M$ matrix U with matrix elements

$$U_{mn} = |\langle a_m | a_n \rangle|^2 \,. \tag{23}$$

These elements obey $U_{nn} = 1$ for $0 \leq n \leq M-1$ and $0 \leq U_{mn} = U_{nm} \leq 1$ for any pair (m, n), with $m \neq n$.

Now we suppose that there is no pair of orthogonal elements, i.e. $0 < U_{mn}$ if $m \neq n$, and no pair of proportional elements, i.e. $U_{mn} < 1$ if $m \neq n$, in the frame. Then from the Perron-Frobenius theorem for (strictly) positive matrices (e.g., [27]), the spectral radius $r = r(U)$ is > 0 and is dominant simple eigenvalue of U. It is proved (Collatz-Wielandt formula) that

$$r = \max_{\mathbf{v}\in\mathcal{N}} \min_{0\leq n\leq M-1\,, v_n\neq 0} \frac{(U\mathbf{v})_n}{v_n} \quad \text{where } \mathcal{N} \stackrel{\text{def}}{=} \{\mathbf{v} \,|\, \mathbf{v} \geq 0 \ \mathbf{v} \neq \mathbf{0}\} \,.$$

There exists a unique vector, $\mathbf{v}_r$, $\|\mathbf{v}_r\| = 1$, which is strictly positive (all components are > 0 and can be interpreted as probabilities) and $U\mathbf{v}_r = r\mathbf{v}_r$. All other eigenvalues α of U lie within the open disk of radius r : $|\alpha| < r$. Since $\mathrm{tr}\, U = M$, and that U has M eigenvalues, one should have $r > 1$. The value $r = 1$ represents precisely the limit case in which all eigenvalues are 1, i.e. $U = \mathbb{I}$ and the frame is just an orthonormal basis of $\mathbb{C}^M$. It is then natural to view the number

$$\eta \stackrel{\text{def}}{=} r - 1 \tag{24}$$

as a kind of "distance" of the frame to the orthonormality. The question is to find the relation between the set $\{\kappa_0, \kappa_1, ..., \kappa_{M-1}\}$ of weights defining the frame and the distance η. By projecting on each vector $|a_n\rangle$ from both sides the frame resolution of the unity (16), we easily obtain the M equations

$$1 = \langle a_m | a_m \rangle = \sum_{n=0}^{M-1} \kappa_n |\langle a_m | a_n \rangle|^2 \,, \quad \text{i.e.} \quad U\mathbf{v}_\kappa = \mathbf{v}_\delta \,, \tag{25}$$

where ${}^t\mathbf{v}_\kappa \stackrel{\text{def}}{=} (\kappa_0\, \kappa_1 \dots \kappa_{M-1})$ and ${}^t\mathbf{v}_\delta \stackrel{\text{def}}{=} (1\,1 \dots 1)$ is the first diagonal vector in $\mathbb{C}^M$. Note that if U is not singular we have $\mathbf{v}_\kappa = U^{-1}\,\mathbf{v}_\delta$. In the "uniform" case for which $\kappa_n = d/M$ for all n, i.e. in the case of a finite equal norm Parseval frame, which means that $\mathbf{v}_\kappa = (d/M)\,\mathbf{v}_\delta$, then the radius $r = M/d$ and $\mathbf{v}_r = 1/\sqrt{M}\,\mathbf{v}_\delta$. In this case, the distance to orthonormality is just

$$\eta = \frac{M-d}{d} \,, \tag{26}$$

a relation which clearly exemplifies what we can expect at the limit $d \to M$.

Let us now examine the matrix $P \stackrel{\text{def}}{=} U\,K$, where $\mathrm{K} \stackrel{\text{def}}{=} \mathrm{diag}(\kappa_0, \kappa_1, \dots, \kappa_{M-1})$. Its (right) stochastic nature is evident from (25). The row vector $\boldsymbol{\varpi} \stackrel{\text{def}}{=} {}^t\mathbf{v}_\kappa/d = \left(\frac{\kappa_0}{d}\,\frac{\kappa_1}{d}\,\cdots\,\frac{\kappa_M - 1}{d}\right)$ is a stationary probability vector:

$$\boldsymbol{\varpi}\, P = \boldsymbol{\varpi} \,. \tag{27}$$

As is well known, this vector obeys the ergodic property:

$$\lim_{k\to\infty}\left(P^k\right)_{mn} = \varpi_n = \frac{\kappa_n}{d}\,. \tag{28}$$

3.3. Semi-classical aspects of finite frame quantization through lower symbols

With the above stochastic matrix approach developed above, the relation between upper and lower symbols,

$$\check{f}(a_n) = \langle a_n|A_f|a_n\rangle = \sum_{m=0}^{M-1}\kappa_m\, f(a_m)\,|\langle a_n|a_m\rangle|^2\,,$$

is rewritten as

$$\check{\mathbf{f}} = P\mathbf{f}\,, \tag{29}$$

with ${}^t\mathbf{f} \stackrel{\mathrm{def}}{=} (f(a_0)\, f(a_1)\, ...\, f(a_{M-1}))$ and ${}^t\check{\mathbf{f}} \stackrel{\mathrm{def}}{=} \left(\check{f}(a_0)\, \check{f}(a_1)\, ...\, \check{f}(a_{M-1})\right)$. This formula is interesting because it can be iterated:

$$\check{\mathbf{f}}^{[k]} = P^k\mathbf{f}\,, \quad \check{\mathbf{f}}^{[k]} = P\check{\mathbf{f}}^{[k-1]}\,, \quad \check{\mathbf{f}}^{[1]} \equiv \check{\mathbf{f}}\,, \tag{30}$$

and so we find from the property (28) of P that the ergodic limit (or "long-term average") of the iteration stabilizes to the "classical" average of the observable f defined as:

$$\check{\mathbf{f}}^{[\infty]} = \langle f\rangle_{\mathrm{cl}}\,\mathbf{v}_\delta\,, \qquad \text{where} \quad \langle f\rangle_{\mathrm{cl}} \stackrel{\mathrm{def}}{=} \sum_{n=0}^{M-1}\frac{\kappa_n}{d} f(a_n)\,. \tag{31}$$

We can conclude from this fact that the chain of maps $P : \check{f}^{[k-1]} \mapsto \check{f}^{[k]}$ corresponds to a sort of increasing regularization of the original function f.

We can evaluate the "distance" between the lower symbol $\check{\mathbf{f}}$ and its classical counterpart f through the inequality:

$$\|\check{\mathbf{f}} - \mathbf{f}\|_\infty \stackrel{\mathrm{def}}{=} \max_{0\leq n\leq M-1}|f(a_n) - \check{f}(a_n)| \leq \|\mathbb{I} - P\|_\infty\,\|\mathbf{f}\|_\infty\,, \tag{32}$$

where the induced norm [31] on matrix A is $\|A\|_\infty = \max_{0\leq m\leq M-1}\sum_{n=0}^{M-1}|a_{mn}|$. In the present case, because of the stochastic nature of P, we have

$$\|\mathbb{I} - P\|_\infty = 2\left(1 - \min_{0\leq n\leq M-1}\kappa_n\right)\,. \tag{33}$$

In the uniform case, $\kappa_n = d/M$ for all n, we thus have an estimate of how far the two functions f and $\check{f}$ are: $\|\check{\mathbf{f}} - \mathbf{f}\|_\infty \leq 2(M-d)/M\,\|\mathbf{f}\|_\infty$. In the general case, we can view the parameter

$$\zeta \stackrel{\mathrm{def}}{=} 1 - \min_{0\leq n\leq M-1}\kappa_n \tag{34}$$

as an alternative "distance" of the "quantum world" to the classical one, of non-commutativity to commutativity, or again of the frame to orthonormal basis, like the distance $\eta = r - 1$ introduced in (24). Note that in the uniform case $\kappa_n = d/M$ these two parameters are simply related:

$$\zeta = 1 - d/M = \frac{\eta}{1+\eta}\,.$$

3.4. Wigner and Weyl functions in the context of frame quantization

The Weyl (or ambiguity) function $\widetilde{W}_f(\alpha,\beta)$ of A_f is

$$\begin{aligned}\widetilde{W}_f(\alpha,\beta) &= \mathrm{Tr}[D(\alpha,\beta)A_f] = \sum_{n=0}^{M-1}\kappa_n\, f(a_n)\langle a_n|D(\alpha,\beta)|a_n\rangle \\ &= \mathrm{e}^{-\frac{\pi\mathrm{i}}{d}\alpha\beta}\sum_{r=0}^{d-1}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}r\beta}\sum_{n=0}^{M-1} f(a_n)\phi_{r-\alpha}(a_n)\overline{\phi_r(a_n)}\,. \end{aligned} \tag{35}$$

The Wigner function $W_f(\alpha,\beta)$ of A_f is

$$\begin{aligned}W_f(\alpha,\beta) &= \mathrm{Tr}[P(\alpha,\beta)A_f] = \sum_{n=0}^{M-1}\kappa_n\, f(a_n)\langle a_n|P(\alpha,\beta)|a_n\rangle \\ &= \mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}\alpha\beta}\sum_{r=0}^{d-1}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}r\beta}\sum_{n=0}^{M-1} f(a_n)\phi_{-r-2\alpha}(a_n)\overline{\phi_r(a_n)}\,. \end{aligned} \tag{36}$$

If f is real, then A_f is a Hermitian operator and the Wigner function is real. Using Eq.(11) we prove that the Weyl and Wigner functions are related through the Fourier transform

$$\widetilde{W}_f(\alpha,\beta) = \frac{1}{d}\sum_{\gamma,\delta}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}(\alpha\delta-\beta\gamma)}W_f(\gamma,\delta). \tag{37}$$

The operator A_f can be written as

$$A_f = \sum_{\alpha,\beta}W_f(\alpha,\beta)P(\alpha,\beta) = \sum_{\alpha,\beta}\widetilde{W}_f(-\alpha,-\beta)D(\alpha,\beta) \tag{38}$$

The proof is based on the expression (21) for the matrix elements of A_f with respect to the orthonormal basis $|e_j\rangle$.

4. Examples of uniform tight frames

4.1. A 'lattice frame'

It is known that some of the formulas are different in odd-dimensional Hilbert spaces from their counterparts in even-dimensional Hilbert spaces. This is also true in the related area of finite sums [8]. In this example, and also in some cases below, we consider the odd-dimensional case.

Let $d=2s+1$ be an odd number. In this case $\mathbb{Z}_d=\{-s,-s+1,...,s-1,s\}$. We consider an example of the general tight frames discussed earlier, where $\mathcal{X}=\mathbb{Z}_d\times\mathbb{Z}_d$ and

$$\phi_k:\mathbb{Z}_d\times\mathbb{Z}_d\longrightarrow\mathbb{C},\qquad \phi_k(\alpha,\beta)=\frac{1}{\sqrt{d}}\,\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}\beta k}\bar{w}_{\alpha+k}\,. \tag{39}$$

Here the w_l's where $l\in\mathbb{Z}_d$ are d complex numbers normalized so that $\sum_{l=-s}^{s}|w_l|^2=1$. The spectral radius of the matrix U is $r=d$ and "distances" of such a frame to orthonormality are respectively $\eta=d-1$ and $\zeta=1-1/d$. This frame is fairly appropriate to our purpose to explore quantum versions of the finite phase space $\mathcal{X}$ obtained through frame quantization.

It is easily proved that the functions ϕ_k form an orthonormal system for the inner product (12). Hence our general formalism leads to the d^2 unit vectors

$$|\alpha,\beta\rangle = \sum_{k=-s}^{s} \mathrm{e}^{\frac{2\pi \mathrm{i}}{d}\beta k} w_{\alpha+k} |e_k\rangle, \qquad (\alpha,\beta)\in\mathbb{Z}_d\times\mathbb{Z}_d\,. \tag{40}$$

They form a finite tight frame in the Hilbert space $\mathcal{H}$ with uniform distribution $\kappa_{\alpha\beta} = d/d^2 = 1/d$ (which means that no point in the phase space $\mathcal{X} = \mathbb{Z}_d\times\mathbb{Z}_d$ is privileged):

$$\frac{1}{d}\sum_{\alpha,\beta=-s}^{s} |\alpha,\beta\rangle\langle\alpha,\beta| = \mathbb{I}_{\mathcal{H}}. \tag{41}$$

The overlap between two elements of the frame is

$$\langle\alpha,\beta|\gamma,\delta\rangle = \sum_{k=-s}^{s} \mathrm{e}^{\frac{2\pi \mathrm{i}}{d}(\delta-\beta)k}\overline{w}_{\alpha+k} w_{\gamma+k} \tag{42}$$

(note that the Fourier transform of $\overline{w}_{\alpha+k}w_{\gamma+k}$ appears here). In Ref.[49] a similar frame is obtained by starting from a discrete version of the Heisenberg group [39, 41, 43] defined in terms of the displacement operators. Then the resolution of the identity of Eq.(41) follows from the general Eq.(9). In this section we derive some of their properties through our approach, rather than through the properties of the displacement operators.

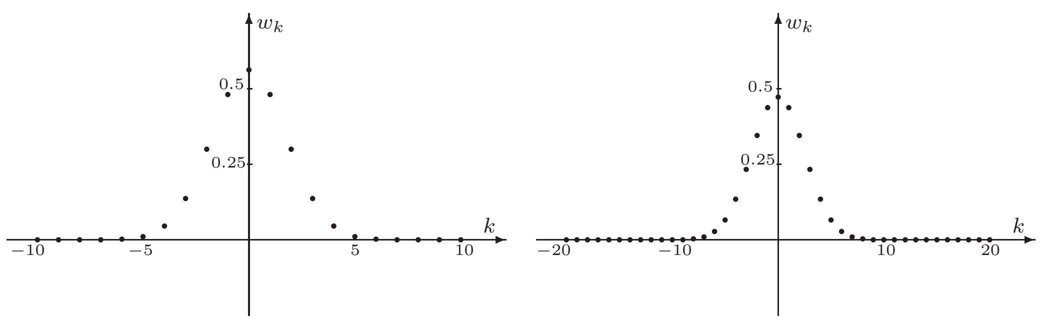

Figure 1. The w_k of Eq.(43) in the cases $d = 21$ (left) and $d = 41$ (right).

We choose as w_k the following numbers (for examples see figure 1):

$$w_k = \frac{1}{\sqrt{\mathcal{N}_0}}\sum_{\alpha=-\infty}^{\infty} \mathrm{e}^{-\frac{\pi}{d}(\alpha d+k)^2}\,; \qquad \mathcal{N}_0 = \sum_{\gamma=-\infty}^{\infty} \mathrm{e}^{-\frac{2\pi}{d}\gamma^2}\,. \tag{43}$$

This is similar to a Zak [50] or Weil [45] transform of the Gaussian function, and

$$w_k = w_{k+d} = w_{-k} \tag{44}$$

for any k. Using the Jacobi theta function

$$\theta_3(z;\tau) = \sum_{j=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\pi\tau j^2 + 2\mathrm{i}jz} \tag{45}$$

we rewrite w_k as

$$w_k = \frac{1}{\sqrt{d}} \frac{\theta_3\left(\frac{\pi k}{d}; \frac{\mathrm{i}}{d}\right)}{\sqrt{\theta_3\left(0; \frac{2\mathrm{i}}{d}\right)}}. \tag{46}$$

The components w_k have significant values only near 0 (modulo d). This is expected because w_k is related to the Gaussian function $\mathrm{e}^{-x^2/2}$ (which is the ground state wave function in the harmonic oscillator formalism). It is known [30] that w_k are the coordinates of an eigenvector of the finite Fourier transform

$$\frac{1}{\sqrt{d}} \sum_{l=-s}^{s} \mathrm{e}^{\frac{2\pi \mathrm{i}}{d} kl} w_l = w_k . \tag{47}$$

We can now prove the following relation

$$F|\alpha, \beta\rangle = \mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}\alpha\beta} |\beta, -\alpha\rangle \tag{48}$$

4.2. A 'sparse frame'

We assume that n is a divisor of d and we consider the following partition of the set $\mathbb{Z}_d$:

$$S_k = \{k, n+k, 2n+k, ..., d-n+k\}; \qquad \bigcup_{k=0}^{n-1} S_k = \mathbb{Z}_d \tag{49}$$

Here $k = 0, ..., n-1$. We also consider the subspaces $\mathcal{H}_k$ of the Hilbert space $\mathcal{H}$, defined as:

$$\mathcal{H}_k = \mathrm{span}\{|e_r\rangle \mid r \in S_k\}; \quad \mathcal{H} = \bigoplus_{k=0}^{n-1} \mathcal{H}_k. \tag{50}$$

We call Π_k the projector to the subspace $\mathcal{H}_k$.

We use the notation $D = d/n$ and assume that n, D are *coprime*. We now give another example of tight frames where $\mathcal{X} = \bigcup(S_k \times S_k)$. Clearly this is a subset of $\mathbb{Z}_d \times \mathbb{Z}_d$, with cardinality d^2/n. Let $|w\rangle$ be a fiducial vector such that

$$\langle w|\Pi_k|w\rangle = \frac{1}{n} \tag{51}$$

We consider the following sets of vectors

$$\mathcal{L}_k = \{|\alpha, \beta; k\rangle = D(\alpha, \beta)|w\rangle \mid \alpha, \beta \in S_k\}; \qquad k = 0, ..., n-1. \tag{52}$$

Each of these sets has $(d/n)^2$ vectors, so their union $\mathcal{L} = \bigcup \mathcal{L}_k$ has d^2/n vectors. In the appendix we show that

$$\frac{n^2}{d} \sum_{\alpha,\beta \in S_k} |\alpha, \beta; k\rangle\langle\alpha, \beta; k| = \Pi_k. \tag{53}$$

The $\Pi_k|\alpha, \beta; k\rangle$ can be viewed as a tight frame within $\mathcal{H}_k$. From this follows that

$$\frac{n^2}{d} \sum_{k=1}^{n-1} \sum_{\alpha,\beta \in S_k} |\alpha, \beta; k\rangle\langle\alpha, \beta; k| = \mathbb{I}_{\mathcal{H}}. \tag{54}$$

and therefore the set $\mathcal{L}$ of vectors, are a tight frame in $\mathcal{H}$.

5. Quantum mechanics for finite systems based on finite frame quantization

Let $d\!=\!2s\!+\!1$ be a positive odd number. The space $\mathcal{X} = \mathbb{Z}_d \times \mathbb{Z}_d$ identified with the set

$$\mathcal{Z}_d = \{-s, -s\!+\!1, ..., s\} \times \{-s, -s\!+\!1, ..., s\} \tag{55}$$

is now considered as a finite version of a phase space, i.e. the mechanical phase space for the motion of a particle on the periodic set $\mathbb{Z}_d$. In accordance with the content of Section 3 we associate to each *classical observable* $f\!:\!\{-s, -s\!+\!1, ..., s\}\!\times\!\{-s, -s\!+\!1, ..., s\} \longrightarrow \mathbb{R}$ the linear operator

$$A_f : \mathcal{H} \longrightarrow \mathcal{H}\,, \qquad A_f = \frac{1}{d} \sum_{\alpha,\beta=-s}^{s} f(\alpha,\beta)\, |\alpha,\beta\rangle\langle\alpha,\beta|\,, \tag{56}$$

and the lower symbol $\check{f}\!=\!\check{A}_f\!:\!\{-s, -s\!+\!1, ..., , s\}\!\times\!\{-s, -s\!+\!1, ..., s\} \longrightarrow \mathbb{R}$

$$\check{f}(a,b) = \langle a,b|A_f|a,b\rangle\,. \tag{57}$$

We know from (32) that the deviation in the sense of the norm $\|\cdot\|_\infty$ is bounded by

$$\|\mathbf{f}-\check{\mathbf{f}}\|_\infty \le 2\zeta\, \|\mathbf{f}\|_\infty\,, \quad \text{with } \zeta = 1-\frac{1}{d}\,. \tag{58}$$

5.1. Example

The linear operators corresponding to the functions

$$\begin{array}{ll} q^\nu\!:\!\{-s, -s\!+\!1, ..., s\}\!\times\!\{-s, -s\!+\!1, ..., s\}\!\longrightarrow\!\mathbb{R}\,, & q^\nu(\alpha,\beta) = \alpha^\nu\,, \\ p^\nu\!:\!\{-s, -s\!+\!1, ..., s\}\!\times\!\{-s, -s\!+\!1, ..., s\}\!\longrightarrow \mathbb{R}\,, & p^\nu(\alpha,\beta) = \beta^\nu \end{array} \tag{59}$$

where $\nu\!\in\!\{1,2,3,...\}$, are $A_{q^\nu},\, A_{p^\nu} : \mathcal{H} \longrightarrow \mathcal{H}$,

$$A_{q^\nu} = \frac{1}{d} \sum_{\alpha,\beta=-s}^{s} \alpha^\nu\, |\alpha,\beta\rangle\langle\alpha,\beta|\,, \qquad A_{p^\nu} = \frac{1}{d} \sum_{\alpha,\beta=-s}^{s} \beta^\nu\, |\alpha,\beta\rangle\langle\alpha,\beta|\,, \tag{60}$$

and they satisfy the relation

$$A_{q^\nu} = F A_{p^\nu} F^+ = -F^+ A_{p^\nu} F. \tag{61}$$

The 'position states' $\{|e_l\rangle\}_{l=-s}^{s}$ are eigenvectors of A_q and the 'momentum states' $\{|f_k\rangle\}_{k=-s}^{s}$ are eigenvectors of A_p. More exactly, we have

$$\begin{array}{l} A_q\, |e_l\rangle = \lambda_l |e_l\rangle \\ A_p\, |f_k\rangle \!=\! \lambda_{-k} |f_k\rangle \!=\! -\lambda_k |f_k\rangle \end{array} \qquad \text{where} \quad \lambda_j = \sum_{\alpha=-s}^{s} \alpha\, w_{\alpha+j}^2\,. \tag{62}$$

Therefore,

$$A_q \!=\! \sum_{l=-s}^{s} \lambda_l\, |e_l\rangle\langle e_l|\,, \qquad A_p \!=\! \sum_{k=-s}^{s} \lambda_{-k}\, |f_k\rangle\langle f_k| \!=\! -\sum_{k=-s}^{s} \lambda_k\, |f_k\rangle\langle f_k|\,. \tag{63}$$

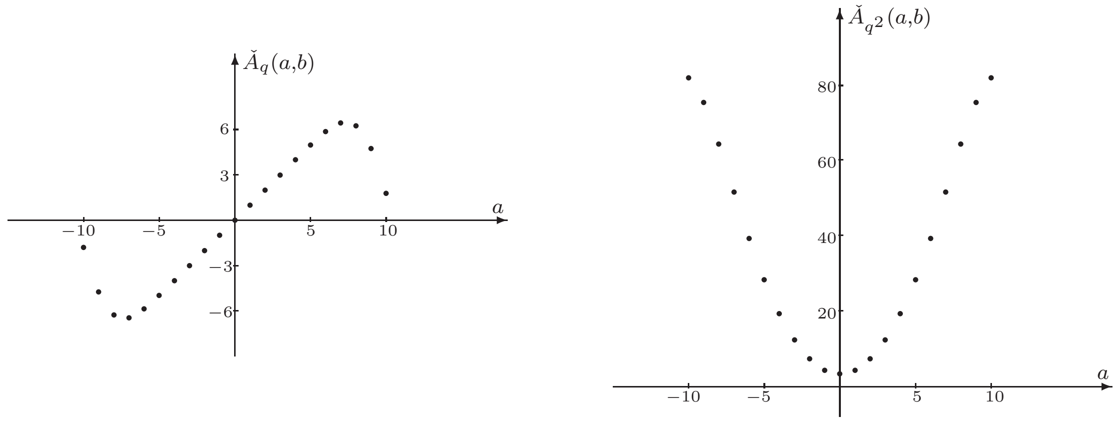

Figure 2. The a-dependence of $\check{A}_q(a,b)$ and $\check{A}_{q^2}(a,b)$ in the case $d=21$ i.e. $M=441$.

As expected the lower symbol $\check{A}_{q^\nu}(a,b)$ does not depend on b

$$\check{A}_{q^\nu}(a,b) = \sum_{k=-s}^{s} w_{a+k}^2 \sum_{\alpha=-s}^{s} \alpha^\nu w_{\alpha+k}^2 \,, \tag{64}$$

and the lower symbol $\check{A}_{p^\nu}(a,b)$ does not depend on a

$$\check{A}_{p^\nu}(a,b) = \sum_{k=-s}^{s} w_{b+k}^2 \sum_{\alpha=-s}^{s} \alpha^\nu w_{\alpha+k}^2 \,. \tag{65}$$

In particular, we have

$$\check{q}(a,b) = \sum_{k=-s}^{s} \lambda_k w_{a+k}^2 \,, \quad \check{p}(a,b) = \sum_{k=-s}^{s} \lambda_k w_{b+k}^2 \,, \tag{66}$$

and

$$\langle a,b|(A_q)^2|a,b\rangle = \sum_{k=-s}^{s} \lambda_k^2 w_{a+k}^2 \,, \qquad \langle a,b|(A_p)^2|a,b\rangle = \sum_{k=-s}^{s} \lambda_k^2 w_{b+k}^2 \,. \tag{67}$$

The a-dependence of $\check{q}(a,b)$ and $\check{A}_{q^2}(a,b)$ in the case $d=21$, i.e. $M=441$, is presented in figure 2. Since $\|\mathbf{q}\|_\infty = d-1 = \|\mathbf{p}\|_\infty$, the bound to deviation (58) gives for the position and momentum:

$$\|\mathbf{q}-\check{\mathbf{q}}\|_\infty \le 2\frac{(d-1)^2}{d} \,, \quad \|\mathbf{p}-\check{\mathbf{p}}\|_\infty \le 2\frac{(d-1)^2}{d} \,, \tag{68}$$

which is clearly not an optimal bound in view of the figure 2 (actually we should think in terms of *relative* deviation).

We can also show that

$$(\Delta A_q)^2 = \langle a,b|(A_q)^2|a,b\rangle - (\langle a,b|A_q|a,b\rangle)^2 = \sum_{k=-s}^{s} \lambda_k \left(\lambda_k - \sum_{l=-s}^{s} \lambda_l w_{a+l}^2\right) w_{a+k}^2 \,. \tag{69}$$

The operators $X, P : \mathcal{H} \longrightarrow \mathcal{H}$,

$$X = \sum_{l=-s}^{s} l\, |e_l\rangle\langle e_l|\,, \qquad P = \sum_{k=-s}^{s} k\, |f_k\rangle\langle f_k|\,, \tag{70}$$

satisfy the relation

$$X = F^{+}PF = -FPF^{+} \tag{71}$$

and the corresponding lower symbols are

$$\begin{aligned} \check{X}(a,b) &= \langle a,b|X|a,b\rangle = \lambda_a \\ \check{P}(a,b) &= \langle a,b|P|a,b\rangle = \lambda_{-b} = -\lambda_b\,. \end{aligned} \tag{72}$$

The distribution of the eigenvalues of X, A_q and the distribution of the eigenvalues of $\frac{1}{2}(P^2{+}X^2)$, $\frac{1}{2}(A_p^2{+}A_q^2)$, $A_{\frac{1}{2}(p^2+q^2)}$ obtained by using the matrix elements

$$\begin{aligned} &\langle e_k|\tfrac{1}{2}(P^2{+}X^2)|e_l\rangle = \tfrac{1}{2}k^2\,\delta_{kl} + \tfrac{1}{2d}\textstyle\sum_{j=-s}^{s} j^2\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}(k-l)j} \\ &\langle e_k|\tfrac{1}{2}(A_p^2{+}A_q^2)|e_l\rangle = \tfrac{1}{2}\lambda_k^2\,\delta_{kl} + \tfrac{1}{2d}\textstyle\sum_{j=-s}^{s} \lambda_j^2\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}(k-l)j} \\ &\langle e_k|A_{\frac{1}{2}(p^2+q^2)}|e_l\rangle = \tfrac{1}{2}\delta_{kl}\textstyle\sum_{\alpha=-s}^{s}\alpha^2 w_{\alpha+k}^2 + \tfrac{1}{2d}\sum_{\alpha,\beta=-s}^{s}\beta^2\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}\beta(k-l)}w_{\alpha+k}w_{\alpha+l} \end{aligned} \tag{73}$$

are presented in figure 3. One can remark a tendency, as $M = d^2$ becomes larger and larger w.r.t. d, to have $\lambda_k = k$, and a tendency of the levels of $A_{\frac{1}{2}(p^2+q^2)}$ to become equidistant levels.

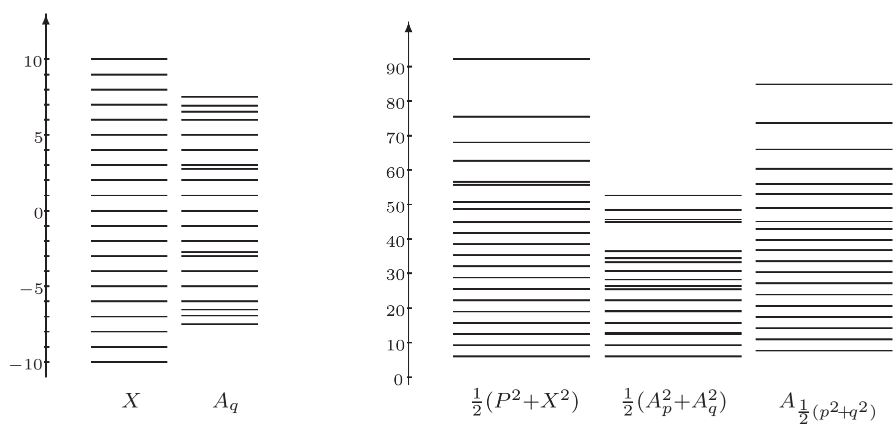

Figure 3. Spectra of X, A_q and $\frac{1}{2}(P^2{+}X^2)$, $\frac{1}{2}(A_p^2{+}A_q^2)$, $A_{\frac{1}{2}(p^2+q^2)}$ in the case $d = 21$.

6. About the non-uniqueness of the upper symbol

The function $f : \mathcal{Z}_d \longrightarrow \mathbb{R}$ is an upper symbol of our position operator X,

$$\frac{1}{d}\sum_{\alpha,\beta=-s}^{s} f(\alpha,\beta)\,|\alpha,\beta\rangle\langle\alpha,\beta| = \sum_{j=-s}^{s} j\,|e_j\rangle\langle e_j|$$

if and only if the values $f(\alpha,\beta)$ satisfy the system of d^2 equations

$$\frac{1}{d}\sum_{\alpha,\beta=-s}^{s} f(\alpha,\beta)\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}\beta(k-l)} w_{\alpha+k}w_{\alpha+l} = k\,\delta_{kl}.$$

Generally the upper symbol corresponding to a linear operator is not unique. Looking for a function f depending only on α, that is, a function of the form $f(\alpha,\beta)=\varphi(\alpha)$ the previous sytem of equations becomes

$$\frac{1}{d}\sum_{\alpha,\beta=-s}^{s} \varphi(\alpha)\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}\beta(k-l)} w_{\alpha+k}w_{\alpha+l} = k\,\delta_{kl}.$$

and is equivalent to

$$\sum_{\alpha=-s}^{s}\varphi(\alpha)\,w_{\alpha+k}^2 = k\,,\qquad k\in\{-s,-s+1,...,s-1,s\}. \tag{74}$$

that is,

$$\left(\begin{array}{lllll} w_{-s}^2 & w_{-s+1}^2 & ... & w_{s-1}^2 & w_s^2 \\ w_{-s+1}^2 & w_{-s+2}^2 & ... & w_s^2 & w_{-s}^2 \\ ... & ... & ... & ... & ... \\ w_s^2 & w_{-s}^2 & ... & w_{s-2}^2 & w_{s-1}^2 \end{array}\right)\left(\begin{array}{c} \varphi(-s) \\ \varphi(-s+1) \\ \vdots \\ \varphi(s) \end{array}\right) = \left(\begin{array}{c} -s \\ -s+1 \\ \vdots \\ s \end{array}\right).$$

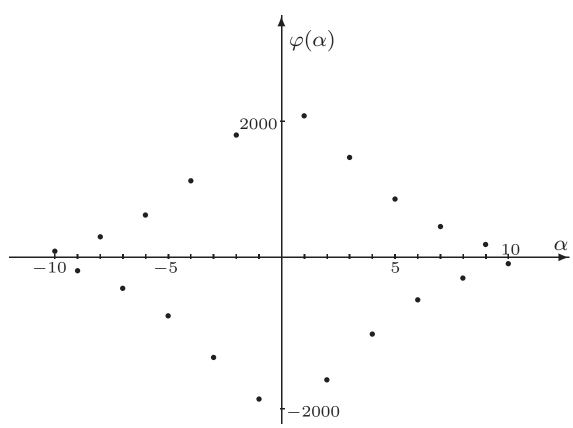

Figure 4. The upper symbol $\varphi(\alpha)$ of X obtained by solving (74) in the case $d=21$.

The matrix of the system is (up to a permutation of rows) a circulant matrix with the absolute value of the derminant

$$\prod_{n=0}^{d-1}\left(w_0^2+\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}n}\;w_1^2+\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}2n}\;w_2^2+\ldots+\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}(d-1)n}\;w_{d-1}^2\right). \tag{75}$$

A discussion of necessary and sufficient conditions for circulant matrices to be nonsingular is presented in [16]. The upper symbol $f(\alpha,\beta)=\varphi(\alpha)$ of the position operator x, obtained by solving (74) in the case $d=21$, is presented in the figure 4.

7. Concluding remarks

The finite-dimensional quantum systems represent an essential ingredient in the development of several fields including spin systems, ensembles of two-level atoms, quantum dots, quantum optics and quantum information [1, 9, 32, 43]. In the present paper we have studied tight frames in this context and related phase space quantities (covariant and contravariant symbols and Wigner and Weyl functions). The general theory has been exemplified with novel tight frames (in section 4) which have been used for calculations of matrix elements, spectra, etc. If we use a different fiducial vector, or if we perform unitary transformations on $|\alpha,\beta\rangle$, we get other tight frames. But it is an interesting problem for further work to give examples of tight frame, that they be uniform or not.

There are several other problems related to finite quantum systems: symplectic transformations and tomography, mutually unbiased bases, multipartite systems comprised of many finite quantum systems and their entanglement, etc. In future works we also plan to approach these questions from a finite tight frame point of view.

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8. Appendix

We consider the matrix elements of both sides of Eq.(53) with $\langle e_r|$ and $|e_{r'}\rangle$, and we get

$$\frac{n^2}{d}\sum_{\alpha,\beta\in S_k}\sum_{m,m'\in\mathbb{Z}_d}\langle e_r|D(\alpha,\beta)|e_m\rangle\langle e_m|s\rangle\langle s|e_{m'}\rangle\langle e_{m'}|[D(\alpha,\beta)]^\dagger|e_{r'}\rangle$$
$$=\delta(r,r') \tag{76}$$

where $r,r'\in S_k$. We then use the relation

$$\langle e_r|D(\alpha,\beta)|e_m\rangle=\exp\left[\frac{i2\pi}{d}\left(-\frac{\alpha\beta}{2}+\beta r\right)\right]\delta(r,m-\alpha) \tag{77}$$

A comment here is that since $r,\alpha\in S_k$, the right hand side can be nonzero only when m belongs to S_{2k}. Now Eq.(76) simplifies to

$$\frac{n^2}{d}\sum_{\alpha,\beta\in S_k}\sum_{m,m'\in\mathbb{Z}_d}\exp\left[\frac{i2\pi}{d}(\beta r-\beta r')\right]\delta(r,m-\alpha)\,\delta(r',m'-\alpha)$$
$$\times\langle e_m|s\rangle\langle s|e_{m'}\rangle=\delta(r,r'). \tag{78}$$

This simplifies further to

$$\frac{n^2}{d}\sum_{\alpha,\beta\in S_k}\exp\left[\frac{i2\pi}{d}(\beta r-\beta r')\right]\langle e_{r+\alpha}|s\rangle\langle s|e_{r'+\alpha}\rangle=\delta(r,r'). \tag{79}$$

We next consider the following subset of $\mathbb{Z}_d$:

$$T=\{0,D,2D,...,(n-1)D\},;\qquad D=\frac{d}{n} \tag{80}$$

and prove that

$$\frac{n}{d}\sum_{\beta\in S_k}\exp\left(i\frac{2\pi\beta\ell}{d}\right)=\sum_{t\in T}\exp\left(i\frac{2\pi kt}{d}\right)\delta(\ell,t). \tag{81}$$

In the special case that $\ell\in S_0$ and n, D are coprime this simplifies to

$$\frac{n}{d}\sum_{\beta\in S_k}\exp\left(i\frac{2\pi\beta\ell}{d}\right)=\delta(\ell,0). \tag{82}$$

This is because ℓ is an integer multiple of n and t is an integer multiple of D. Therefore ℓ can be equal to t only if $\ell=t=0$ (the case $\ell=t=Dn$ is the same because $Dn=d=0(\text{mod }d)$). In Eq.(79) $r-r'\in S_0$, and therefore it simplifies to

$$n\sum_{\alpha\in S_k}\delta(r-r',0)\;\langle e_{r+\alpha}|s\rangle\langle s|e_{r'+\alpha}\rangle=\delta(r,r'). \tag{83}$$

But for fiducial vectors which obey the constraint of Eq.(51), this is true. This proves Eq.(53).

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