Fourth order interference and the extended phase space formalism for finite quantum systems

H. Al Hadhrami, A. Vourdas
Department of Computing,
University of Bradford,
Bradford BD7 1DP, United Kingdom

Finite quantum systems where the position x and the momentum p take values in $\mathbb{Z}(d)$ are considered. An extended phase-space $x-p-X-P$, where X is position increment and P is momentum increment, is introduced. The extended Wigner function $\mathcal{W}(x,p,X,P)$ and the extended Weyl function $\widetilde{\mathcal{W}}(x',p',X',P')$ are defined and their properties are studied. They quantify fourth order interference. Several examples are discussed.

I. INTRODUCTION

There is a lot of work on the Wigner function $W(x,p)$ and the Weyl function $\widetilde{W}(X,P)$ for the harmonic oscillator. Here x,p are the position and momentum and X,P are position increment and momentum increment. The Wigner function shows the quantum noise in the state and the Weyl function is a generalized correlation function (it shows the overlap between the displaced function and the original function).

When two functions $f(q_1,...,q_N)$ and $F(Q_1,...,Q_N)$ are related through a Fourier trasform (in any context), we can introduce the $2N$-dimensional phase space $q_i - Q_i$ and in it we can define Wigner and Weyl functions. Since the Wigner and Weyl functions are related through a two-dimensional Fourier transform, the concept of the extended phase space $x-p-X-P$ has been considered in refs[1, 2]. An extended Wigner function $\mathcal{W}(x,p,X,P)$ has been introduced, which is a quartic function of the wavefunction and which shows fourth order interferences. It has been shown that there are uncertainty relations for $\delta x \delta P$ and also for $\delta P \delta x$ which for pure states are related to the usual uncertainties Δx, Δp, but for mixed states they are very different. The formalism provides a deeper insight into the connection between quantum noise and quantum correlations.

In this paper we extend this work to quantum systems where the position and momentum take values in $\mathbb{Z}(d)$ (the integers modulo d) and the Hilbert space is finite dimensional (reviews with extensive lists of references have been presented in [3]). Recently there has been a lot of work on Wigner and Weyl functions for such systems[3–13]. Here we introduce the extended phase space $x-p-X-P$ where $x,p,X,P \in \mathbb{Z}(d)$. In it we define the extended Wigner function $\mathcal{W}(x,p,X,P)$ and the extended Weyl function $\widetilde{\mathcal{W}}(x',p',X',P')$, and study their properties.

In section II we first state without proof known facts from the theory of finite quantum systems, in order to establish the notation [3]. We then introduce the Legendre states, which are related to the Legendre symbol [14]. In section III we discuss briefly the Wigner function $W(x,p)$ and the Weyl function $\widetilde{W}(X,P)$ for finite quantum systems. They are related to second order interference.

In section IV we introduce the extended Wigner function $\mathcal{W}(x,p,X,P)$ and the extended Weyl function $\widetilde{\mathcal{W}}(x',p',X',P')$. We explain that they are related to fourth order interference and discuss their marginal properties. We also present numerical results for several examples.

We conclude in section V with a discussion of our results.

II. FINITE QUANTUM SYSTEMS

A finite quantum system with a d-dimensional Hilbert space is considered. There are problems in defining Wigner and Weyl functions in the 'Fermi case' that d is an even number [3], and for this reason in this paper we consider the 'Bose case' that d is an odd number.

Let $|\mathcal{X};m\rangle$ be an orthonormal basis of 'position states'. Here $\mathcal{X}$ is not a variable, it simply indicates position states. m is a variable and takes values in $\mathbb{Z}(d)$. The Fourier operator is defined as:

$$F = d^{-1/2}\sum_{m,n\in\mathbb{Z}(d)}\omega(mn)|\mathcal{X};m\rangle\langle\mathcal{X};n|;\qquad \omega(m)=\exp\left[i\frac{2\pi m}{d}\right] \tag{1}$$

Another orthonormal basis, the 'momentum states' is defined as

$$|\mathcal{P};m\rangle = F|\mathcal{X};m\rangle = d^{-1/2}\sum_{n\in\mathbb{Z}(d)}\omega(mn)|\mathcal{X};n\rangle;\qquad m\in\mathbb{Z}(d) \tag{2}$$

Here $\mathcal{P}$ is not a variable but it indicates momentum states.

Throughout the paper $\hbar=\Omega=1$ where Ω is a unit for momenta and Ω^{-1} is a unit for positions. The momentum is the ratio of the physical momentum over Ω and it is a dimensionless number in $\mathbb{Z}(d)$, and the position is the physical position times Ω and it is a dimensionless number in $\mathbb{Z}(d)$. All quantities in the figures below, are dimensionless.

An important relation which is used in the proof of many formulas below, is

$$\frac{1}{d}\sum_{n\in\mathbb{Z}(d)}\omega[n(m-\ell)] = \delta(m,\ell) \tag{3}$$

where $\delta(m,\ell)$ is Kronecker's delta (it is equal to 1 when $m=\ell$ modulo d).

The position-momentum phase space is the toroidal lattice $\mathbb{Z}(d)\times\mathbb{Z}(d)$. In it we define the displacement operators

$$Z^{\alpha}|\mathcal{P};m\rangle = |\mathcal{P};m+\alpha\rangle,\qquad Z^{\alpha}|\mathcal{X};m\rangle = \omega(\alpha m)|X;m\rangle. \tag{4}$$

$$X^{\beta}|\mathcal{X};m\rangle = |\mathcal{X};m+\beta\rangle,\qquad X^{\beta}|\mathcal{P};m\rangle = \omega(-\beta m)|\mathcal{P};m\rangle. \tag{5}$$

where $\alpha,\beta\in\mathbb{Z}(d)$. The general displacement operators are defined as

$$D(\alpha,\beta) = Z^{\alpha}X^{\beta}\omega(-2^{-1}\alpha\beta) \tag{6}$$

We note that since d is an odd number the 2^{-1} is an integer in $\mathbb{Z}(d)$ (if $d=2j+1$ then $2^{-1}=j+1$). The operators $D(\alpha,\beta)\omega(\gamma)$ form a representation of the Heisenberg-Weyl group. Acting with Fourier transform on the displacement operator we get:

$$FD(\alpha,\beta)F^{\dagger} = D(\beta,-\alpha) \tag{7}$$

The parity operator Π_0 around the origin, is defined as

$$\Pi_0|\mathcal{X};m\rangle = |\mathcal{X};-m\rangle;\qquad \Pi_0|\mathcal{P};m\rangle = |\mathcal{P};-m\rangle \tag{8}$$

The parity operator $\Pi(x,p)$ around the point $(x,p)\in\mathbb{Z}(d)\times\mathbb{Z}(d)$, is defined as

$$\Pi(x,p) \;=\; D(x,p)\Pi_0D^{\dagger}(x,p) = D(2x,2p)\Pi_0 = \Pi_0D(-2x,-2p) \tag{9}$$

A. Eigenstates of the Fourier operator

In the harmonic oscillator context the eigenstates of the Fourier operator $\exp(i\pi\hat{n}/2)$, where $\hat{n}$ is the number operator, are the number states. In the present context the Fourier operator is a $d \times d$ matrix and has d eigenvectors. The fact that $F^4 = \mathbf{1}$ shows that the eigenvalues are $1, i, -1, -i$. It is known[15] that for $d = 4m + 1$ the multiplicity of these eigenvalues is $m+1, m, m, m$, and for $d = 4m+3$ the multiplicity of these eigenvalues is $m+1, m+1, m+1, m$ correspondingly.

In the case that the dimension d is an odd prime number p, there is an eigenvector of the Fourier operator F, which is related to the Legendre symbol [14] $(n|p)$:

- $(n|p) = 0$ if $n = 0 \pmod p$;
- $(n|p) = 1$ if $n \neq 0 \pmod p$ and there exists x such that $x^2 = n \pmod p$;
- $(n|p) = -1$ if $n \neq 0 \pmod p$ and there is no x such that $x^2 = n \pmod p$;

It is well known that

$$(n|p) = n^{(p-1)/2} \pmod p \tag{10}$$

We define the Legendre states as

$$|L\rangle = (p-1)^{-1/2} \sum_{n=0}^{p-1} (n|p) |\mathcal{X}; n\rangle \tag{11}$$

Similar states have been used for quantum algorithms in [16]. We can show that

$$F|L\rangle \;=\; \lambda|L\rangle; \qquad \lambda = p^{-1/2} \sum_{n=1}^{p-1} (n|p)\omega(n) \tag{12}$$

It is a well known result in the theory of Gauss sums[14] that $\lambda = 1$ if $p = 4k + 1$ and $\lambda = i$ if $p = 4k + 3$.

We also define the displaced number states

$$|L; \alpha, \beta\rangle = D(\alpha, \beta)|L\rangle = (p-1)^{-1/2} \sum_{n=0}^{p-1} (n|p)\; \omega(2^{-1}\alpha\beta + \alpha n)\; |\mathcal{X}; n + \beta\rangle \tag{13}$$

where $\alpha, \beta \in \mathbb{Z}(p)$. Using Eqs.(7), (12) we show that

$$F|L; \alpha, \beta\rangle = \lambda|L; \beta, -\alpha\rangle \tag{14}$$

We next show that

$$\frac{1}{p} \sum_{\alpha,\beta} |L; \alpha, \beta\rangle\langle L; \alpha, \beta| = \mathbf{1} \tag{15}$$

This is a special case of a general relation

$$\frac{1}{p} \sum_{\alpha,\beta} D(\alpha, \beta) \frac{\Theta}{\mathrm{Tr}\Theta} [D(\alpha, \beta)]^\dagger = \mathbf{1} \tag{16}$$

which is valid for an arbitrary operator Θ[3].

III. WIGNER AND WEYL FUNCTIONS

Let Θ be an arbitrary operator with matrix elements

$$\Theta_X(m,n) \equiv \langle \mathcal{X}; m|\Theta|\mathcal{X}; n\rangle, \qquad \Theta_P(m,n) \equiv \langle \mathcal{P}; m|\Theta|\mathcal{P}; n\rangle \tag{17}$$

Its Wigner function $W(x,p)$ where $x,p \in \mathbb{Z}(d)$, is defined as:

$$\begin{aligned} W_\Theta(x,p) &= \mathrm{Tr}[\Theta\Pi(x,p)] = \omega(2xp) \sum_{\ell\in\mathbb{Z}(d)} \omega(-2p\ell)\ \Theta_X(\ell, 2x-\ell) \\ &= \omega(-2xp) \sum_{\ell\in\mathbb{Z}(d)} \omega(2x\ell)\ \Theta_P(\ell, 2p-\ell) \end{aligned} \tag{18}$$

It is easily seen that if $\Theta = \alpha_1\Theta_1 + \alpha_2\Theta_2$ where α_1, α_2 are complex numbers, then

$$W_\Theta(x,p) = \alpha_1 W_{\Theta_1}(x,p) + \alpha_2 W_{\Theta_2}(x,p) \tag{19}$$

Its Weyl function $\widetilde{W}(X,P)$ where $X, P \in \mathbb{Z}(d)$ are position increment and momentum increment, is defined as

$$\begin{aligned} \widetilde{W}_\Theta(X,P) &= \mathrm{Tr}[\Theta D(X,P)] = \omega(2^{-1}XP) \sum_{\ell\in\mathbb{Z}(d)} \omega(P\ell)\ \Theta_X(\ell, X+\ell) \\ &= \omega(-2^{-1}XP) \sum_{\ell\in\mathbb{Z}(d)} \omega(-X\ell)\ \Theta_P(\ell, P+\ell) \end{aligned} \tag{20}$$

The Weyl function is related to the Wigner function through a two-dimensional Fourier transform.

$$\widetilde{W}_\Theta(X,P) = \frac{1}{d} \sum_{x,p\in\mathbb{Z}(d)} W_\Theta(x,p)\ \omega(Px - Xp) \tag{21}$$

The Wigner function is related to the Weyl function through the inverse Fourier transform:

$$W_\Theta(x,p) = \frac{1}{d} \sum_{X,P\in\mathbb{Z}(d)} \widetilde{W}_\Theta(X,P)\ \omega(-Px + Xp) \tag{22}$$

We next consider the operator $\Theta = |\mathcal{X}; n\rangle\langle\mathcal{X}; m|$. The corresponding Wigner and Weyl functions are

$$\begin{aligned} S_{mn}(x,p) &= \langle \mathcal{X}; m|\Pi(x,p)|\mathcal{X}; n\rangle = \omega(2xp - 2pn)\ \delta(m, 2x-n) \\ \widetilde{S}_{mn}(X,P) &= \langle \mathcal{X}; m|D(X,P)|\mathcal{X}; n\rangle = \omega(2^{-1}XP + Pn)\ \delta(m, n+X) \end{aligned} \tag{23}$$

We can show that $[S_{mn}(x,p)]^* = S_{nm}(x,p)$. The $S_{mn}(x,p)$ form an orthonormal basis in the Hilbert space of functions of two variables. Indeed we show that

$$\begin{aligned} \frac{1}{d}\sum_{x,p} [S_{mn}(x,p)]^*\ S_{m'n'}(x,p) &= \delta(m,m')\ \delta(n,n') \\ \frac{1}{d}\sum_{m,n} [S_{mn}(x,p)]^*\ S_{mn}(x',p') &= \delta(x,x')\ \delta(p,p') \end{aligned} \tag{24}$$

The first of Eqs(24) is proved by multiplying Eq.(23) by its complex conjugate and then perform the summation over x, p, using Eq.(3). The second part of Eqs(24) is proved in a similar way. The $\widetilde{S}_{mn}(X, P)$ also obey similar relations.

The Wigner and Weyl functions of an arbitrary operator Θ can be expressed in terms of the $S_{mn}(x, p)$ and $\widetilde{S}_{mn}(X, P)$ as

$$W_\Theta(x, p) = \sum_{m,n} \Theta_X(n, m)\; S_{mn}(x, p); \qquad \widetilde{W}_\Theta(X, P) = \sum_{m,n} \Theta_X(n, m)\; \widetilde{S}_{mn}(X, P) \tag{25}$$

The marginal properties of both the Wigner and Weyl functions have been discussed in [3].

A. Second order interference

We consider the superposition of two orthogonal states

$$|s\rangle = \alpha_1 |s_1\rangle + \alpha_2 |s_2\rangle; \qquad |\alpha_1|^2 + |\alpha_2|^2 = 1 \tag{26}$$

which is descibed with the density matrix

$$\begin{aligned} \rho &= |\alpha_1|^2 \rho_1 + |\alpha_2|^2 \rho_2 + \sigma; \qquad \rho_i = |s_i\rangle\langle s_i| \\ \rho_m &= |\alpha_1|^2 \rho_1 + |\alpha_2|^2 \rho_2; \qquad \sigma = \alpha_1 \alpha_2^* |s_1\rangle\langle s_2| + \alpha_1^* \alpha_2 |s_2\rangle\langle s_1| \end{aligned} \tag{27}$$

ρ_m is the density matrix of a mixed state (the index 'm' in the notation indicates mixed state). σ describes the second order interference in the superposition of quantum states in Eq.(26)

Eq.(19) shows that the Wigner function can be writen as

$$\begin{aligned} W_\rho(x, p) &= W_{\rho_m}(x, p) + W_\sigma(x, p) \\ W_{\rho_m}(x, p) &= |\alpha_1|^2 W_{\rho_1}(x, p) + |\alpha_2|^2 W_{\rho_2}(x, p) \end{aligned} \tag{28}$$

where $W_\sigma(x, p)$ is the part of the Wigner function which describes second order interference in the superposition of quantum states in Eq.(26)

Similar comments can be made for the Weyl function.

B. Examples

All our examples are for the case $d = 11$. We consider the mixed state

$$\rho_m = \frac{1}{2}(|\mathcal{X}; 0\rangle\langle\mathcal{X}; 0| + |\mathcal{X}; 8\rangle\langle\mathcal{X}; 8|) \tag{29}$$

We also consider the pure state

$$|s\rangle = 2^{-1/2}(|\mathcal{X}; 0\rangle + |\mathcal{X}; 8\rangle) \tag{30}$$

described with the density matrix

$$\rho = \rho_m + \sigma; \qquad \sigma = \frac{1}{2}(|\mathcal{X};0\rangle\langle\mathcal{X};8| + |\mathcal{X};8\rangle\langle\mathcal{X};0|) \tag{31}$$

In figs.1,2 we plot the Wigner function for the pure state of Eq.(30) and the mixed state of Eq.(29). Comparison of the two figures shows that the second order interference terms, which are related to the matrix σ in Eq.(31), are at $x = 4$ in fig.1.

In figs.3,4 we plot the absolute value of the Weyl function for the pure state of Eq.(30) and the mixed state of Eq.(29). From these two figures it is seen that the second order interference terms, which are related to the matrix σ in Eq.(31), are at $X = 3$ and $X = 8$ in fig.3.

In fig.5, we plot the Wigner function for the Legendre state of Eq.(11).

IV. WIGNER AND WEYL FUNCTIONS IN EXTENDED PHASE SPACE

The extended Wigner function of a pair of operators (Θ_1, Θ_2) is defined as

$$\begin{aligned}\mathcal{W}_{(\Theta_1,\Theta_2)}(x,p,X,P) &\equiv \omega(2xP - 2pX) \sum_{k,\ell\in\mathbb{Z}(d)} \omega(2\ell X - 2kP)\,[W_{\Theta_1}(k,\ell)]^*\,W_{\Theta_2}(2x-k, 2p-\ell) \\ &= \omega(2Xp - 2Px) \sum_{k,\ell\in\mathbb{Z}(d)} \omega(2\ell x - 2kp)\,[\widetilde{W}_{\Theta_1}(k,\ell)]^*\,\widetilde{W}_{\Theta_2}(2X-k, 2P-\ell)\end{aligned} \tag{32}$$

where $x, p, X, P \in \mathbb{Z}(d)$. In order to prove the above equality we substitute the Wigner functions with the Weyl functions using Eq.(22). Then we change the variables k, l into $2x-k, 2p-l$ respectively and perform the summation over X, P using Eq.(3).

The corresponding extended Weyl function is given by

$$\begin{aligned}\widetilde{\mathcal{W}}_{(\Theta_1,\Theta_2)}(x',p',X',P') &\equiv \omega(2^{-1}x'P' - 2^{-1}X'p') \sum_{k,\ell\in\mathbb{Z}(d)} \omega(kP' - \ell X')\,[W_{\Theta_1}(k,\ell)]^*\,W_{\Theta_2}(x'+k, p'+\ell) \\ &= \omega(2^{-1}X'p' - 2^{-1}P'x') \sum_{k,\ell\in\mathbb{Z}(d)} \omega(kp' - \ell x')\,[\widetilde{W}_{\Theta_1}(k,\ell)]^*\,\widetilde{W}_{\Theta_2}(X'+k, P'+\ell)\end{aligned} \tag{33}$$

where $x', p', X', P' \in \mathbb{Z}(d)$. The equality is proved in a similar way as the equality in Eq.(32).

We can show that the extended Weyl function is the four-dimensional Fourier transform of the extended Wigner function:

$$\widetilde{\mathcal{W}}_{(\Theta_1,\Theta_2)}(x',p',X',P') = \frac{1}{d^2} \sum_{x,p,X,P} \mathcal{W}_{(\Theta_1,\Theta_2)}(x,p,X,P)\,\omega(Xp' - X'p + xP' - x'P) \tag{34}$$

In order to prove this we insert Eq.(32) into Eq.(34) and perform the summation using Eq.(3). We note that the variables x', p', X', P' are dual to the variables P, X, p, x, correspondingly.

We next show that:

$$[\mathcal{W}_{(\Theta_1,\Theta_2)}(x,p,X,P)]^* = \mathcal{W}_{(\Theta_2,\Theta_1)}(x,p,X,P) \tag{35}$$

In order to prove this we take the complex conjugate of Eq.(32), we change the variables k, l into $2x - k', 2p - \ell'$ and perform the summation over k', ℓ', using Eq.(3).

Most of the properties below will be for the 'extended auto-Wigner' function, i.e., for the case for which $\Theta_1 = \Theta_2 = \Theta$. In this case we use the simpler notation $\mathcal{W}_\Theta(x, p, X, P)$ (and similarly for the Weyl function). From Eq.(35) it follows that $\mathcal{W}_\Theta(x, p, X, P)$ is real. We also prove that

$$[\widetilde{\mathcal{W}}_\Theta(x', p', X', P')]^* = \widetilde{\mathcal{W}}_\Theta(-x', -p', -X', -P') \tag{36}$$

In order to prove this we take the complex conjugate of Eq.(33), we change the variables k, l into $x' + k', p' + \ell'$ and perform the summation over k', ℓ', using Eq.(3).

In the special case $\Theta = |\mathcal{X}; n\rangle\langle\mathcal{X}; m|$, we easily show that

$$\begin{aligned}
\mathcal{S}_{mnm'n'}(x, p, X, P) &\equiv \omega(2xP - 2pX) \sum_{k,\ell \in \mathbb{Z}(d)} \omega(2\ell X - 2kP)\, [S_{mn}(k, \ell)]^*\, S_{m'n'}(2x - k, 2p - \ell) \\
&= \omega(8xp + 2xP - 2pX)\omega(-(m + n)(P + 2p) - 4pn') \\
&\times\ \delta(X - 2x + n + n', 0)\, \delta(4x - m - m' - n - n', 0) \\
\widetilde{\mathcal{S}}_{mnm'n'}(x', p', X', P') &\equiv \omega(2^{-1}x'P' - 2^{-1}X'p') \sum_{k,\ell \in \mathbb{Z}(d)} \omega(kP' - \ell X')\, [S_{mn}(k, \ell)]^*\, S_{m'n'}(x' + k, p' + \ell) \\
&= \omega(2x'p' + 2^{-1}x'P' - 2^{-1}p'X')\omega((m + n)(2^{-1}P' + p') - 2p'n') \\
&\times\ \delta(-X' + x' + n - n', 0)\, \delta(2x' + m - m' + n - n', 0)
\end{aligned} \tag{37}$$

Using these relations, we express the extended Wigner function of an operator Θ with matrix elements given in Eq.(17), as

$$\mathcal{W}_\Theta(x, p, X, P) = \sum_{m,n,m',n'} [\Theta_X(n, m)]^*\, \Theta_X(n', m')\, \mathcal{S}_{mnm'n'}(x, p, X, P) \tag{38}$$

Similarly we express the extended Weyl function of Θ as

$$\widetilde{\mathcal{W}}_\Theta(x', p', X', P') = \sum_{m,n,m',n'} [\Theta_X(n, m)]^*\, \Theta_X(n', m')\, \widetilde{\mathcal{S}}_{mnm'n'}(x', p'X', P') \tag{39}$$

A. Fourth order interference

We consider the mixed state described by the density matrix ρ_m in Eq.(27). Inserting $W_{\rho_m}(x, p)$ from Eq.(28) into Eq.(32) we prove that the extended Wigner function can be written as

$$\begin{aligned}
\mathcal{W}_{\rho_m}(x, p, X, P) &= |\alpha_1|^4 \mathcal{W}_{\rho_1}(x, p, X, P) + |\alpha_2|^4 \mathcal{W}_{\rho_2}(x, p, X, P) + \mathcal{U}_{\rho_m}(x, p, X, P) \\
\mathcal{U}_{\rho_m}(x, p, X, P) &= 2|\alpha_1|^2|\alpha_2|^2 \Re[\mathcal{W}_{(\rho_1,\rho_2)}(x, p, X, P)]
\end{aligned} \tag{40}$$

We next consider the pure state described by the density matrix ρ in Eq.(27). Inserting $W_\rho(x,p)$ from Eq.(28) into Eq.(32) we prove that the extended Wigner function can be written as

$$\begin{aligned}
\mathcal{W}_\rho(x,p,X,P) &= |\alpha_1|^4\mathcal{W}_{\rho_1}(x,p,X,P)+|\alpha_2|^4\mathcal{W}_{\rho_2}(x,p,X,P)+\mathcal{V}_{\rho_m}(x,p,X,P)\\
\mathcal{V}_{\rho_m}(x,p,X,P) &= \mathcal{U}_{\rho_m}(x,p,X,P)\\
&+ 2|\alpha_1|^2\Re[\mathcal{W}_{(\rho_1,\sigma)}(x,p,X,P)]+2|\alpha_2|^2\Re[\mathcal{W}_{(\rho_2,\sigma)}(x,p,X,P)]\\
&+ \mathcal{W}_\sigma(x,p,X,P)
\end{aligned} \tag{41}$$

where σ has been given in Eq.(27). It is seen that the extended Wigner functions of both ρ_m and ρ have fourth order interference terms. The $\mathcal{W}_{\rho_m}(x,p,X,P)$ has the $\mathcal{U}_{\rho_m}(x,p,X,P)$. The $\mathcal{W}_\rho(x,p,X,P)$ has the $\mathcal{V}_{\rho_m}(x,p,X,P)$ which is equal to $\mathcal{U}_{\rho_m}(x,p,X,P)$ plus some extra terms.

B. Marginal properties

In the following context we represent some properties of the extended Wigner function in Eq.(32) and the extended Weyl function in Eq(33) where we consider systems with odd dimension d. We refer to the Wigner and Weyl functions in Eqs.(18),(20). For the extended Wigner function we got:

$$\begin{aligned}
\frac{1}{d^2}\sum_{X,P}\mathcal{W}_\Theta(x,p,X,P) &= |W_\Theta(x,p)|^2\\
\frac{1}{d^2}\sum_{x,p}\mathcal{W}_\Theta(x,p,X,P) &= |\widetilde{W}_\Theta(X,P)|^2\\
\frac{1}{d^3}\sum_{x,p,X,P}\mathcal{W}_\Theta(x,p,X,P) &= \mathrm{Tr}[\Theta\Theta^\dagger]
\end{aligned} \tag{42}$$

In order to prove these relations, we use Eqs.(32) and perform the summation over the variables, using Eq.(3). The proof of the last of these equations requires the relation [3]

$$\frac{1}{d}\sum_{x,p}|W_\Theta(x,p)|^2=\frac{1}{d}\sum_{x,p}|\widetilde{W}_\Theta(X,P)|^2=\mathrm{Tr}[\Theta^\dagger\Theta] \tag{43}$$

In a similar way we prove the following relations for the extended Weyl function

$$\begin{aligned}
\frac{1}{d^2}\sum_{X',P'}\widetilde{\mathcal{W}}_\Theta(x',p',X',P') &= [W_\Theta(-2^{-1}x',-2^{-1}p']^*\;W_\Theta(2^{-1}x',2^{-1}p')\\
\frac{1}{d^2}\sum_{x',p'}\widetilde{\mathcal{W}}_\Theta(x',p',X',P') &= [\widetilde{W}_\Theta(-2^{-1}X',-2^{-1}P')]^*\;\widetilde{W}_\Theta(2^{-1}X',2^{-1}P')\\
\frac{1}{d^2}\sum_{x',p',X',P'}\widetilde{\mathcal{W}}_\Theta(x',p',X',P') &= \mathcal{W}_\Theta(0,0,0,0)
\end{aligned} \tag{44}$$

We have explained earlier that for odd d, the inverse of 2 in $\mathbb{Z}(d)$ exists.

We can also prove the following properties which include squares of the extended Wigner functions

$$\begin{aligned}\frac{1}{d^2}\sum_{X,P}[\mathcal{W}_\Theta(x,p,X,P)]^2 &= \sum_{k,\ell}|\ [W_\Theta(k,\ell)]^*\ W_\Theta(2x-k,2p-\ell)|^2\\ \frac{1}{d^2}\sum_{x,p}[\mathcal{W}_\Theta(x,p,X,P)]^2 &= \sum_{k,\ell}|\ [\widetilde{W}_\Theta(k,\ell)]^*\ \widetilde{W}_\Theta(2X-k,2P-\ell)|^2\\ \frac{1}{d^4}\sum_{x,p,X,P}[\mathcal{W}_\Theta(x,p,X,P)]^2 &= [\mathrm{Tr}(\Theta^\dagger\Theta)]^2\end{aligned} \tag{45}$$

They are proved if we multiply Eq.(32) with itself and perform the summation using Eq.(3). The proof of the last of these equations requires Eq.(43).

In a similar way we prove the following relations that involve the squares of the extended Weyl function:

$$\begin{aligned}\frac{1}{d^2}\sum_{X',P'}|\widetilde{\mathcal{W}}_\Theta(x',p',X',P')|^2 &= \sum_{k,\ell}|\ [W_\Theta(k,\ell)]^*\ W_\Theta(x'+k,p'+\ell)|^2\\ \frac{1}{d^2}\sum_{x',p'}|\widetilde{\mathcal{W}}_\Theta(x',p',X',P')|^2 &= \sum_{k,\ell}|\ [\widetilde{W}_\Theta(k,\ell)]^*\ \widetilde{W}_\Theta(X'+k,P'+\ell)|^2\\ \frac{1}{d^4}\sum_{x',p',X',P'}|\widetilde{\mathcal{W}}_\Theta(x',p',X',P')|^2 &= [\mathrm{Tr}(\Theta^\dagger\Theta)]^2\end{aligned} \tag{46}$$

C. Examples

In figs.6,7 we plot the extended Wigner function $\mathcal{W}_\rho(x,p,0,0)$ for the pure state of Eq.(30) and the mixed state of Eq.(29), correspondingly. Comparison of these figures shows that in both cases we have fourth order interference terms, which in both figures are located at $x = 4$. They are different in the two cases as has been explained in Eqs(40),(41).

The $\mathcal{W}_\rho(0,0,X,P)$ is the same for both the pure state of Eq.(30) and the mixed state of Eq.(29), and it is shown in fig. 8. It is seen that in the example that we have considered, fourth order interference terms are zero, when $x = p = 0$.

In Figures 9,10 we present the extended Wigner function $\mathcal{W}_\rho(x,p,0,0)$ and $\mathcal{W}_\rho(0,0,X,P)$ for the Legendre state.

V. DISCUSSION

We have extended recent work on phase space methods for finite quantum systems, by introducing the concept of the extended phase space $x-p-X-P$ where $x,p,X,P \in \mathbb{Z}(d)$. In Eqs(32),(33), we have introduced the extended Wigner function $\mathcal{W}(x,p,X,P)$ and the extended Weyl function $\widetilde{\mathcal{W}}(x',p',X',P')$ which are related through the Fourier transform of Eq.(34). They obey the marginal relations in Eqs.(42),(44),(45),(46). From a physical point of view, we have explained that they are related to fourth order interference.

In the example of Eq.(27), we have considered a pure state described with the density matrix ρ and a mixed state described with the density matrix ρ_m, where the off-diagonal terms are absent. We have seen in Eq.(41), that the extended Wigner function has fourth order interference terms in both of these cases, although in the case of the pure state there are some additional terms.

The concept of the extended phase space can provide a deeper understanding to interferences in finite quantum systems.

VI. ACKNOWLEDGEMENT

Helpful discussions with Prof. R.F. Bishop (Manchester) on the extended phase space formalism are gratefully acknowledged.

[1] S. Chountasis, A. Vourdas A., Phys. Rev. A58, 1794 (1998)
S. Chountasis S.,A. Vourdas, J. Phys. A: Math. Gen. 32, 6949(1999)
Chong C., Vourdas A.,J. Phys. A: Math. Gen. 34, 9849 (2001)
A. Vourdas, Phys. Rev. A69, 022108 (2004)
[2] S.A. Ponomarenko and E. Wolf, Phys. Rev. A 63,062106 (2001)
S. Franke-Arnold, G. Huyet, and S.M. Barnett, J. Phys. B 34, 945 (2001)
G.S. Agarwal and S.A. Ponomarenko, Phys. Rev. A 67, 032103 (2003).
[3] A. Vourdas ,Rep. Prog. Phys. 67, 1 (2004)
A. Vourdas , J. Phys. A40, R285 (2007)
[4] W. Wootters, Ann. Phys. (NY), 176, 1 (1987)
K. Gibbons, M.J. Hoffman, W. Wootters, Phys. Rev. A70, 062101 (2004)
[5] D. Galetti, A.F.R. de Toledo-Piza Physica 149A (1988) 267
[6] H. Figueroa, J.M. Gracia-Bondia, J.C. Varilly, J. Math. Phys. 31, 2664 (1990) O. Cohendet, P. Combe, M. Sirugue-Collin, J. Phys. A23, 2001 (1990)
[7] T. Lulek, Rep. Math. Phys. 34, 71 (1994) J. Tolar, G. Hadzitaskos, J. Phys. A30, 2509 (1997)
[8] U. Leonhardt, Phys. Rev. Lett. 74, 4101 (1995)
U. Leonhardt, Phys. Rev. A53, 2998 (1996)
[9] N.M. Atakishiyev, S.M. Chumakov, K.B. Wolf, J. Math. Phys. 39, 6247 (1998)
N.M. Atakishiyev, G.S. Pogosyan, L.E. Vincent, K.B. Wolf, J. Phys. A34, 9381 (2001)
K.B. Wolf, G. Krotzsch, J. Opt. Soc. Am 24, 2871 (2007)
[10] A. Vourdas, J. Phys. A29, 4275 (1996)
A. Vourdas, J. Phys. A38, 8453 (2005)
[11] A. Luis, J. Perina, J. Phys. A31, 1423 (1998)
[12] J.P. Paz, Phys. Rev A65, 062311 (2002)
C. Miquel, J.P. Paz, M. Saraceno, M. Knill, R. Laflamme, C. Negreverdne, Nature 418, 59 (2002)
E.F. Calvao, Phys. Rev. A71, 042302 (2005)
A.O. Pittenger, M.H. Rubin, J. Phys. A38, 6005 (2005)
J.P. Paz, A.J. Roncaglia, M. Saraceno, Phys. Rev. A72, 012309 (2005)
C. Cormick, E.F. Calvao, D. Gottesman, J.P. Paz, A.O. Pittenger, Phys. Rev. A73, 012301 (2006)
C. Cormick, J.P. Paz, Phys. Rev. A74, 062315 (2006)

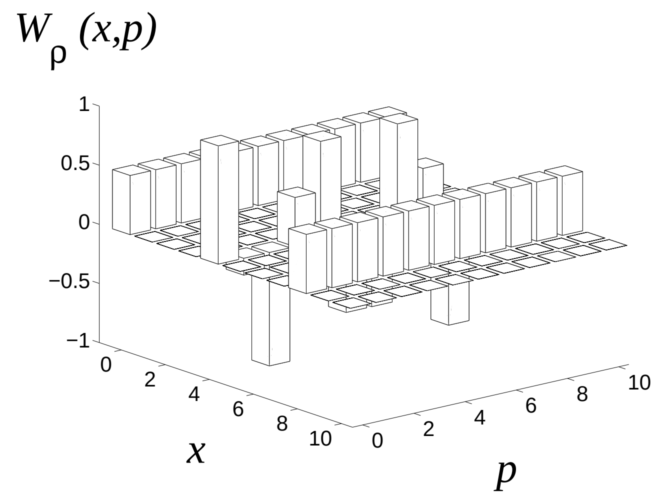

FIG. 1: The Wigner function $W_\rho(x,p)$ as a function of x, p, for the pure state of Eq.(30).

[13] A.B. Klimov, C. Munoz, J.L. Romero, J. Phys. A39, 14471 (2006)
G. Bjork, J.L. Romero, A.B. Klimov, L.L. Sanchez-Soto, J. Opt. Soc. Amer. B24, 371 (2007)

[14] B.C. Berndt, R.J. Evans, K.S. Williams, 'Gauss and Jacobi sums' (Wiley, New York, 1998)
A. Terras, 'Fourier analysis on finite groups and applications' (London Math. Soc. London, 1999)
T.M. Apostol, 'Introduction to analytic number theory' (Springr, Berlin, 1976)

[15] L. Auslander, R. Tolimieri, Bull. Am. Math. Soc. 1, 847 (1979)

[16] W. van Dam, S. Hallgren, SIAM J. Comp. 36, 763 (2006)

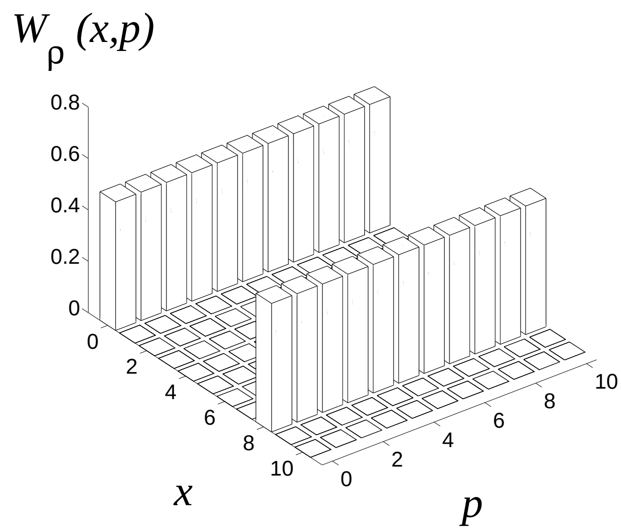

FIG. 2: The Wigner function $W_\rho(x,p)$ as a function of x, p, for the mixed state of Eq.(29).

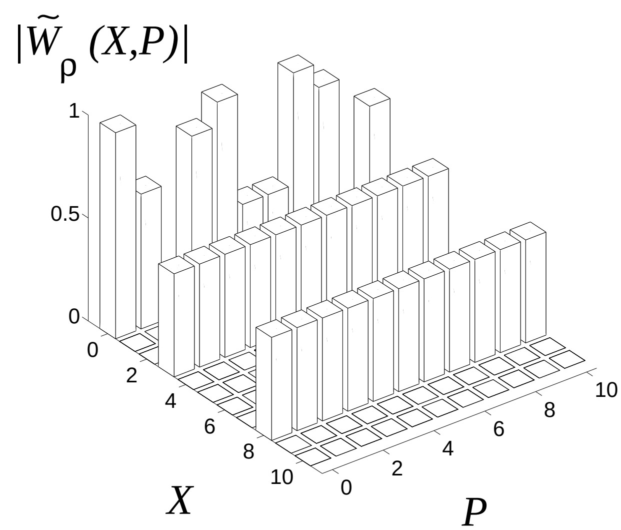

FIG. 3: The absolute value of the Weyl function $|\widetilde{W}_\rho(X,P)|$ as a function of X, P, for the pure state of Eq.(30).

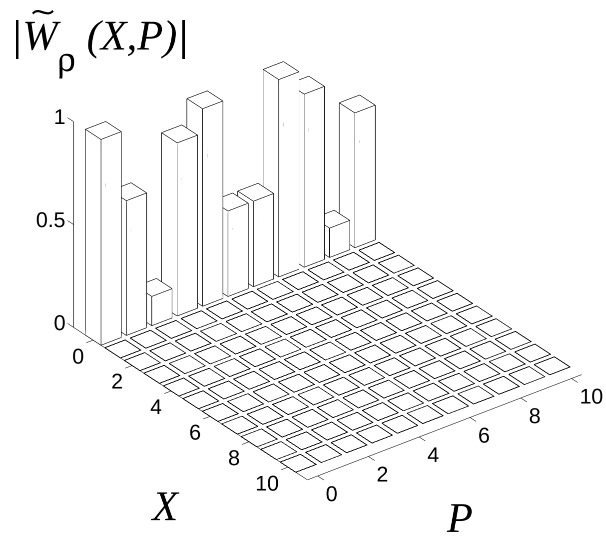

FIG. 4: The absolute value of the Weyl function $|\widetilde{W}_\rho(X, P)|$ as a function of X, P, for the mixed state of Eq.(29).

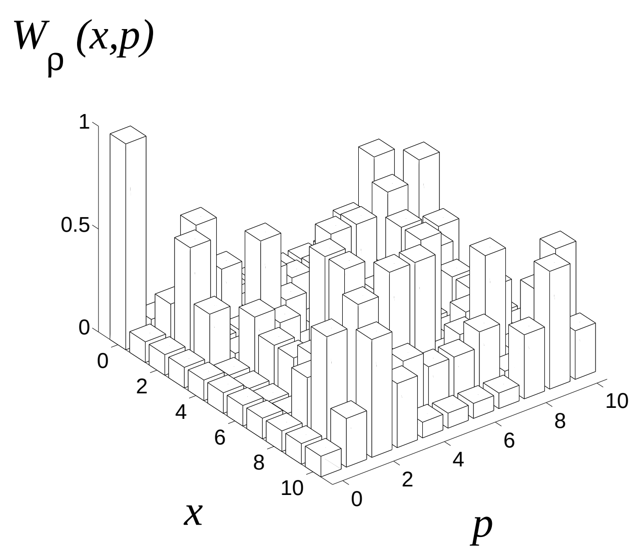

FIG. 5: The Wigner function $W_\rho(x, p)$ as a function of x, p, for the Legendre state of Eq.(11).

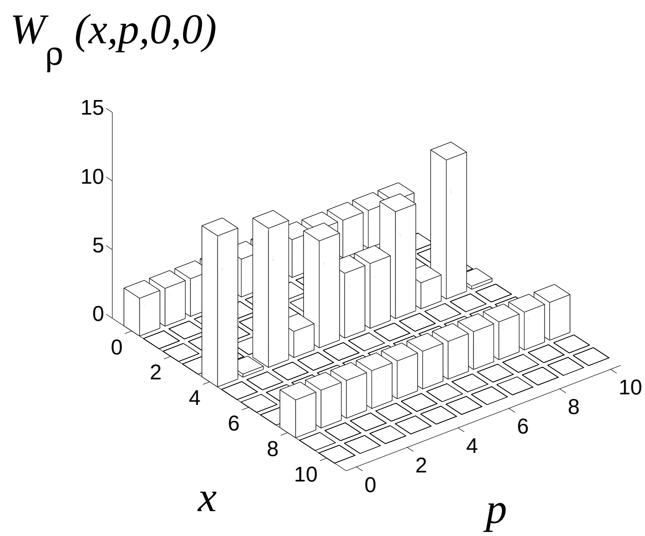

FIG. 6: The extended Wigner function $\mathcal{W}_\rho(x, p, 0, 0)$ as a function of x, p, for the pure state of Eq.(30).

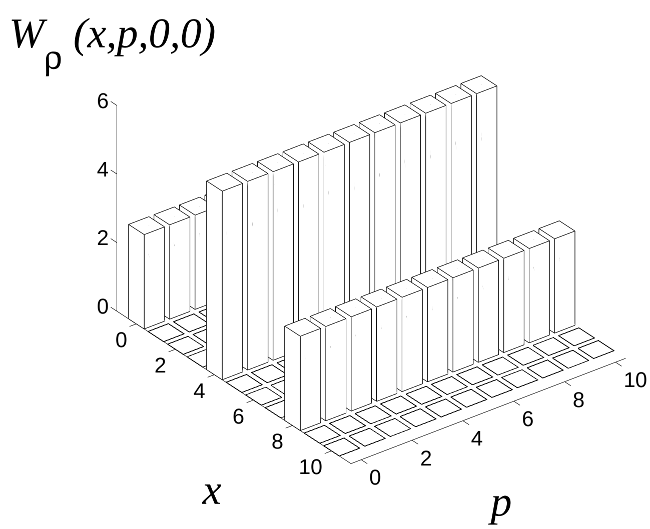

FIG. 7: The extended Wigner function $\mathcal{W}_\rho(x, p, 0, 0)$ as a function of x, p, for the mixed state of Eq.(29).

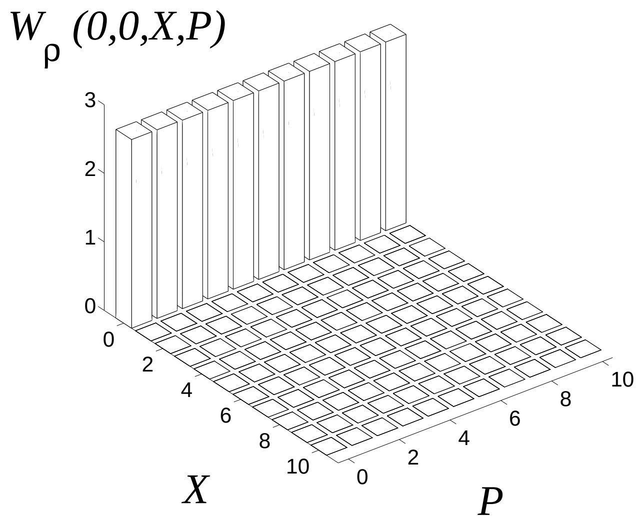

FIG. 8: The extended Wigner function $\mathcal{W}_\rho(0,0,X,P)$ as a function of X, P, for the pure state of Eq.(30) and also for the mixed state of Eq.(29).

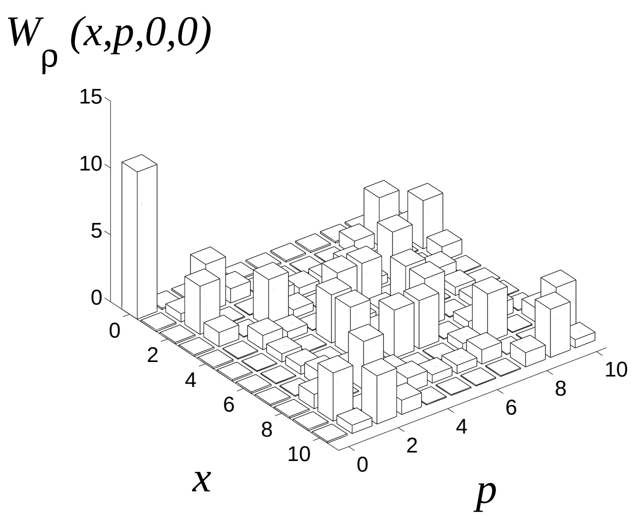

FIG. 9: The extended Wigner function $\mathcal{W}_\rho(x,p,0,0)$ as a function of x, p, for the Legendre state of Eq.(11).

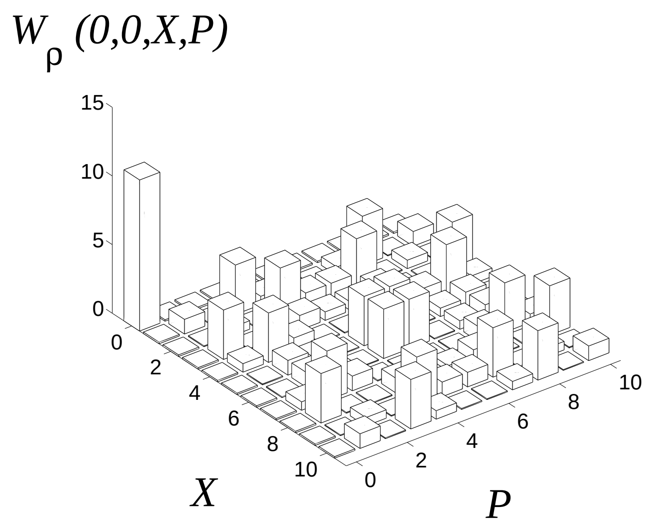

FIG. 10: The extended Wigner function $\mathcal{W}_\rho(0,0,X,P)$ as a function of X, P, for the Legendre state of Eq.(11).