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Totally disconnected and locally compact Heisenberg-Weyl groups

A. Vourdas
Department of Computing,
University of Bradford,
Bradford BD7 1DP, United Kingdom

Harmonic analysis on $\mathbb{Z}(p^\ell)$ and the corresponding representation of the Heisenberg-Weyl group $HW[\mathbb{Z}(p^\ell), \mathbb{Z}(p^\ell), \mathbb{Z}(p^\ell)]$, is studied. It is shown that the $HW[\mathbb{Z}(p^\ell), \mathbb{Z}(p^\ell), \mathbb{Z}(p^\ell)]$ with a homomorphism between them, form an inverse system which has as inverse limit the profinite representation of the Heisenberg-Weyl group $\mathfrak{HW}[\mathbb{Z}_p, \mathbb{Z}_p, \mathbb{Z}_p]$. Harmonic analysis on $\mathbb{Z}_p$ is also studied. The corresponding representation of the Heisenberg-Weyl group $\mathsf{HW}[(\mathbb{Q}_p/\mathbb{Z}_p), \mathbb{Z}_p, (\mathbb{Q}_p/\mathbb{Z}_p)]$ is a totally disconnected and locally compact topological group.

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I. INTRODUCTION

Fourier transforms and harmonic analysis on the ring $\mathbb{Z}(d)$ (the integers modulo d) have been studied by many authors in various contexts. Reviews of this work have been presented in [28, 29]. When d is equal to a prime number p, $\mathbb{Z}(p)$ is a field and we get stronger results (e.g., there are well defined symplectic transformations[7, 21]). The next step in this direction is to use field extension from $\mathbb{Z}(p)$ to the Galois field $GF(p^\ell)$, and study harmonic analysis on $GF(p^\ell)$[30, 31]. The Heisenberg-Weyl groups in these contexts are discrete.

There exists also much work on the Heisenberg-Weyl group with variables in $\mathbb{R}$ (the real numbers), in which case the Heisenberg-Weyl group is continuous. The present work lies in the middle (between discrete and continuous) and studies a profinite and also a totally disconnected and locally compact Heisenberg-Weyl group.

In section II we review briefly aspects of p-adic numbers. The material of this section is known but we explain the notation and define various maps which are used later. In section III we define the Heisenberg-Weyl group. In section IV we consider harmonic analysis on $\mathbb{Z}(p^\ell)$ and the corresponding representations of the Heisenberg-Weyl group $HW[\mathbb{Z}(p^\ell), \mathbb{Z}(p^\ell), \mathbb{Z}(p^\ell)]$ (for all positive integers ℓ). A homomorphism between them is introduced in section V and it is shown that they form an inverse system which has as inverse limit the profinite [24, 38] representation of the Heisenberg-Weyl group $\mathfrak{HW}[\mathbb{Z}_p, \mathbb{Z}_p, \mathbb{Z}_p]$. In this case, the two subgroups $\mathfrak{HW}_1[\mathbb{Z}_p]$ and $\mathfrak{HW}_2[\mathbb{Z}_p]$ (defined below) associated with displacements in the two dual variables, are not Pontryagin dual to each other, and therefore this representation is not applicable to harmonic analysis. This section is a contribution to representations of the Heisenberg-Weyl group.

Harmonic analysis on $\mathbb{Z}_p$ is discussed in section VI. It is shown that the corresponding representation of the Heisenberg-Weyl group $\mathsf{HW}[(\mathbb{Q}_p/\mathbb{Z}_p), \mathbb{Z}_p, (\mathbb{Q}_p/\mathbb{Z}_p)]$ is totally disconnected and locally compact[37] topological group. Some properties of its elements are studied.

Throughout the paper p is an odd prime number. This is needed when we discuss the marginal properties of the displacement operators (Prop.IV.6 and Theo.VI.11 below).

The work is relevant to the general area of harmonic analysis on locally compact fields (e.g. [4, 9, 10, 22, 27, 35, 36]); to work on wavelets on locally compact fields[1, 2, 6, 8, 13, 17–19] and to work on quantum mechanics on p-adic numbers [5, 12, 14–16, 23, 26, 32, 33]. Most of this work studies Fourier transforms on $\mathbb{Q}_p$ which is relevant to the Heisenberg-Weyl group $HW[\mathbb{Q}_p, \mathbb{Q}_p, \mathbb{Q}_p]$. In this paper we present complementary work, on the profinite representation of the Heisenberg-Weyl group $\mathfrak{HW}[\mathbb{Z}_p, \mathbb{Z}_p, \mathbb{Z}_p]$ (which we introduce as an inverse limit), and also on the totally disconnected and locally compact representation of the Heisenberg-Weyl group $\mathsf{HW}[(\mathbb{Q}_p/\mathbb{Z}_p), \mathbb{Z}_p, (\mathbb{Q}_p/\mathbb{Z}_p)]$.

II. PRELIMINARIES

$\mathbb{Q}_p$ is the field of p-adic numbers. Its elements can be written as

$$\alpha = \sum_{\nu=k}^{\infty} \hat{\alpha}_\nu p^\nu; \qquad 0 \le \hat{\alpha}_\nu \le p-1, \tag{1}$$

where k is an integer called ordinal or valuation of α and we denote it as $\mathrm{ord}(\alpha)$. Addition and multiplication is the usual addition and multiplication of series together with the 'carry' operation. In other words, if

$$\alpha = \sum_{\nu=\mathrm{ord}(\alpha)}^{\infty} \hat{\alpha}_\nu p^\nu; \qquad \beta = \sum_{\nu=\mathrm{ord}(\beta)}^{\infty} \hat{\beta}_\nu p^\nu, \tag{2}$$

then

$$\alpha + \beta = \sum_{\nu=\ell}^{\infty} \hat{c}_\nu p^\nu; \qquad \ell = \min(\mathrm{ord}(\alpha), \mathrm{ord}(\beta)), \tag{3}$$

where c_ν can be found from the relations

$$\sum_{\nu=\mathrm{ord}(\alpha)}^{n} \hat{\alpha}_\nu p^\nu + \sum_{\nu=\mathrm{ord}(\beta)}^{n} \hat{\beta}_\nu p^\nu = \sum_{\nu=\ell}^{\infty} \hat{c}_\nu p^\nu \pmod{p^{n+1}}, \tag{4}$$

for all n.

The additive inverse of 1, which we denote as -1, is

$$-1 = (p-1) + (p-1)p + (p-1)p^2 + \dots. \tag{5}$$

$\mathbb{Q}_p$ is a locally compact field of characteristic zero. Topologically it is totally disconnected, Hausdorff topological space. The metric is non-Archimedian, i.e., in addition to the usual properties we also have

$$|\alpha + \beta| \le \max(|\alpha|, |\beta|). \tag{6}$$

The absolute value of α is $|\alpha| = p^{-\mathrm{ord}(\alpha)}$. A neighbourhood of α is the set of p-adic numbers

$$S_n = \left\{ \beta = \sum_{\nu=\mathrm{ord}(\alpha)}^{n} \hat{\alpha}_\nu p^\nu + \sum_{\nu=n+1}^{\infty} \hat{\beta}_\nu p^\nu \right\}. \tag{7}$$

Indeed for all $\beta \in S_n$, we have $|\beta - \alpha| < p^{-n}$.

We can also introduce $\mathbb{Q}_p$ as follows (e.g., chapter 3 in [11]). Given a prime number p, an element a in the field $\mathbb{Q}$ of rational numbers, can be written as

$$a = p^k \frac{b}{c}; \quad a, b, c \in \mathbb{Q}, \tag{8}$$

where any two of the p, b, c are coprime. Here k is the ordinal or valuation of a (chapter 2 in [20], or p.23 in [11]). We define its p-adic absolute value as $|a| = p^{-k}$. Then $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to this absolute value.

$\mathbb{Z}_p$ is the ring of integers in $\mathbb{Q}_p$. We denote as $p^\mu \mathbb{Z}_p$ its principal ideals

$$p^\mu \mathbb{Z}_p = \{\alpha \in \mathbb{Q}_p \,|\, |\alpha| \le p^{-\mu}\} = \{\hat{\alpha}_\mu p^\mu + \hat{\alpha}_{\mu+1} p^{\mu+1} + \hat{\alpha}_{\mu+2} p^{\mu+2} + \dots\}; \qquad \mu \ge 0. \tag{9}$$

A. Complex functions on $\mathbb{Q}_p$ and their integrals

Definition II.1. A complex function $f(x)$ where $x \in \mathbb{Q}_p$ has compact support with degree k, if there exists an integer k such that

$$f(x) = 0 \text{ for } |x| > p^k. \tag{10}$$

Definition II.2. A complex function $f(x)$ where $x \in \mathbb{Q}_p$ is locally constant with degree n, if there exists an integer n such that

$$f(x + \alpha) = f(x) \text{ for } |\alpha| \le p^{-n}. \tag{11}$$

We consider the Schwartz-Bruhat space of locally constant complex functions $f(x)$ (where $x \in \mathbb{Q}_p$) with compact support. In integrals of these functions we use the Haar measure, normalized as:

$$\int_{\mathbb{Z}_p} dx = 1. \tag{12}$$

In this paper we use integrals of functions over $\mathbb{Z}_p$, and also over $\mathbb{Q}_p/\mathbb{Z}_p$ (the Pontryagin dual of $\mathbb{Z}_p$).

Functions over $\mathbb{Z}_p$ have constant support with degree $k \ge 0$. In addition to that we require that they are locally constant functions with degree n. Then their integrals are given by

$$\int_{\mathbb{Z}_p} f(x)dx = p^{-n} \sum f(\hat{x}_0 + \hat{x}_1 p + ... + \hat{x}_{n-1} p^{n-1}), \tag{13}$$

where the summation is over all $\{\hat{x}_0, ..., \hat{x}_{n-1}\}$ $(0 \le \hat{x}_i \le p-1)$.

Let $\mathfrak{p}$ be an element of $\mathbb{Q}_p/\mathbb{Z}_p$. $\mathfrak{p}$ is a coset and throughout the paper we represent it with the element which has integer part equal to zero:

$$\mathfrak{p} = \sum_{\nu=\mathrm{ord}(\mathfrak{p})}^{-1} \hat{\mathfrak{p}}_\nu p^\nu; \qquad 0 \le \hat{\mathfrak{p}}_\nu \le p-1. \tag{14}$$

The product $x\mathfrak{p}$ where $x \in \mathbb{Z}_p$ and $\mathfrak{p} \in \mathbb{Q}_p/\mathbb{Z}_p$ is also a coset in $\mathbb{Q}_p/\mathbb{Z}_p$ and we represent it with the element which has integer part equal to zero.

Functions $f(\mathfrak{p})$ (where $\mathfrak{p} \in \mathbb{Q}_p/\mathbb{Z}_p$) are locally constant with degree $k \ge 0$. In addition to that we require that they have compact support with degree n, and then their integrals are given by:

$$\int_{\mathbb{Q}_p/\mathbb{Z}_p} f(\mathfrak{p})d\mathfrak{p} = \sum f(\hat{\mathfrak{p}}_{-k} p^{-k} + \hat{\mathfrak{p}}_{-k+1} p^{-k+1} + ... + \hat{\mathfrak{p}}_{-1} p^{-1}), \tag{15}$$

where $k = -\mathrm{ord}(\mathfrak{p})$ and the summation is over all $\{\hat{\mathfrak{p}}_{-k}, \hat{\mathfrak{p}}_{-k+1}, ..., \hat{\mathfrak{p}}_{-1}\}$. The counting measure is used here.

Remark II.3. $\mathbb{Q}_p/\mathbb{Z}_p$ are p-adic numbers modulo p-adic integers. Therefore functions over $\mathbb{Q}_p/\mathbb{Z}_p$ can be regarded as functions $f(\alpha)$ over $\mathbb{Q}_p$ which are periodic:

$$f(\alpha + 1) = f(\alpha); \qquad \alpha \in \mathbb{Q}_p. \tag{16}$$

Then integration of $f(\alpha)$ over $\mathbb{Q}_p$ can be written as

$$\int_{\mathbb{Q}_p} d\alpha f(\alpha) = \int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{p} \int_{\mathbb{Z}_p} dx\, f(\mathfrak{p} + x) = \int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{p} f(\mathfrak{p}) \int_{\mathbb{Z}_p} dx = \int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{p} f(\mathfrak{p}). \tag{17}$$

The counting measure on $\mathbb{Q}_p/\mathbb{Z}_p$ ensures that the above relation holds (see ref[1]). Eq.(17) can be regarded as a generalization of the Zak [39] or Weil transform [35] which takes functions from the real line $\mathbb{R}$ to the circle $\mathbb{R}/\mathbb{Z}$. Here we go from $\mathbb{Q}_p$ to $\mathbb{Q}_p/\mathbb{Z}_p$. The summation over all $N \in \mathbb{Z}$ in the Zak transform, corresponds to the integral $\int_{\mathbb{Z}_p} dx$ in Eq.(17). The periodicity of Eq.(16), implies that for $x \in \mathbb{Z}_p$ the $f(\mathfrak{p} + x) = f(\mathfrak{p})$ and then we get the right hand side of Eq.(17).

B. Additive characters

An additive character $\chi(\alpha)$ of $\alpha = \sum \hat{\alpha}_\nu p^\nu$ (where $0 \leq \hat{\alpha}_\nu \leq p-1$) is given by

$$\begin{aligned} \mathrm{ord}(\alpha) \leq -1 \quad &\rightarrow \quad \chi(\alpha) = \exp\left(i2\pi \sum_{\nu=\mathrm{ord}(\alpha)}^{-1} \hat{\alpha}_\nu p^\nu\right) \\ \mathrm{ord}(\alpha) \geq 0 \quad &\rightarrow \quad \chi(\alpha) = 1. \end{aligned} \tag{18}$$

It is a locally constant function of degree $n = 0$. The orthogonality of the characters is expressed in the present context as:

$$\int_{\mathbb{Z}_p} \chi(x\mathfrak{p} - x\mathfrak{p}')dx = \Delta(\mathfrak{p} - \mathfrak{p}'); \qquad \mathfrak{p}, \mathfrak{p}' \in \mathbb{Q}_p/\mathbb{Z}_p, \tag{19}$$

where $\Delta(\mathfrak{p}) = 1$ if $\mathfrak{p} = 0$ and $\Delta(\mathfrak{p}) = 0$ if $\mathfrak{p} \neq 0$.

C. $\mathbb{Z}_p$ as inverse limit

Let $\mathbb{Z}^+$ be the set of positive integers with the usual order. The pair $(\mathbb{Z}^+, \leq)$ is a directed set. The indices in all the inverse and direct limits considered in this paper belong to this directed set. For this reason we do not mention explicitly the set of indices, in every inverse and direct limit that we consider.

The following proposition is well-known (e.g., p.65 in [11]; p.24 in [22]; p.26 in [24]; p.27 in [38]; section 1.4 in [25]):

Proposition II.4. *We consider the $\mathbb{Z}(p^\ell)$ as additive topological groups with the discrete topology. For $k \leq \ell$ we define the 'take remainder' homomorphisms*

$$\varphi_{k\ell} : \ \mathbb{Z}(p^k) \ \leftarrow \ \mathbb{Z}(p^\ell); \qquad k \leq \ell \tag{20}$$

where

$$\varphi_{k\ell}(\alpha_\ell) = \alpha_k; \qquad \alpha_\ell = \alpha_k (\mathrm{mod}\ p^k); \qquad \alpha_\ell \in \mathbb{Z}(p^\ell); \qquad \alpha_k \in \mathbb{Z}(p^k). \tag{21}$$

Then $\{\mathbb{Z}(p^\ell), \varphi_{k\ell}\}$, for $\ell \in \mathbb{Z}^+$, is an inverse system with inverse limit the ring $\mathbb{Z}_p$ of p-adic integers:

$$\varprojlim \mathbb{Z}(p^\ell) = \mathbb{Z}_p. \tag{22}$$

Proof. $\varphi_{k\ell}$ are continuous maps, because their domains have the discrete topology, and if $k \leq \ell \leq r$, then

$$\varphi_{k\ell}[\varphi_{\ell r}(\alpha_r)] = \varphi_{kr}(\alpha_r); \qquad k \leq \ell \leq r; \qquad \alpha_k \in \mathbb{Z}(p^k). \tag{23}$$

In addition to that, the map $\varphi_{\ell\ell}$ is the identity and therefore the set of all $\{\mathbb{Z}(p^\ell), \varphi_{k\ell}\}$ is an inverse system. We mention for later use that it is a surjective inverse system.

The elements of the inverse limit of this inverse system are the sequences

$$\alpha = \{\alpha_1, \alpha_2, ...\}; \qquad \alpha_\ell \in \mathbb{Z}(p^\ell), \tag{24}$$

where $\alpha_k = \varphi_{k\ell}(\alpha_\ell)$ for $k \leq \ell$. Addition and multiplication are performed componentwise. We express the elements of $\mathbb{Z}(p^\ell)$ in base p

$$\alpha_\ell = \sum_{\nu=0}^{\ell-1} \hat{\alpha}_\nu p^\nu; \qquad 0 \leq \hat{\alpha}_\nu \leq p-1. \tag{25}$$

Then the map $\varphi_{k\ell}$ of eq.(21) can be written as

$$\varphi_{k\ell}\left(\sum_{\nu=0}^{\ell-1} \hat{\alpha}_\nu p^\nu\right) = \sum_{\nu=0}^{k-1} \hat{\alpha}_\nu p^\nu; \qquad k \leq \ell. \tag{26}$$

Therefore we can rewrite the elements of the inverse limit as

$$\alpha = \sum_{\nu=0}^{\infty} \hat{\alpha}_\nu p^\nu; \qquad 0 \le \hat{\alpha}_\nu \le p-1. \tag{27}$$

The componentwise addition and multiplication of the elements in Eq.(24), become the addition and multiplication rule of series together with the 'carry' operation. Therefore the inverse limit is isomorphic to $\mathbb{Z}_p$. □

Remark II.5 (projections). The natural projection homomorphism ϖ_ℓ from $\mathbb{Z}_p$ to $\mathbb{Z}(p^\ell)$ is

$$\varpi_\ell(\alpha) = \alpha_\ell = \sum_{\nu=0}^{\ell-1} \hat{\alpha}_\nu p^\nu; \qquad 0 \le \hat{\alpha}_\nu \le p-1. \tag{28}$$

The projections are compatible:

$$k \le \ell \quad \rightarrow \quad \varphi_{k\ell} \circ \varpi_\ell = \varpi_k. \tag{29}$$

Remark II.6 (topology). $\mathbb{Z}_p$ is the inverse limit of an inverse system of finite groups, which we take as having the discrete topology. Therefore $\mathbb{Z}_p$ is a profinite group and as such it is a Hausdorff compact totally disconnected topological group. The family

$$\mathbb{Z}_p \supset p\mathbb{Z}_p \supset p^2\mathbb{Z}_p \supset ... \tag{30}$$

is a fundamental system of neighbourhoods of 0.

D. $\mathbb{Q}_p/\mathbb{Z}_p$ as a direct limit

Definition II.7. The multiplicative cyclic group $C(p^\ell)$ is

$$C(p^\ell) = \{\chi_\ell(\alpha_\ell) | \alpha_\ell \in \mathbb{Z}(p^\ell)\}, \tag{31}$$

where

$$\chi_\ell(\alpha_\ell) = \exp\left(\frac{i2\pi\alpha_\ell}{p^\ell}\right); \qquad \alpha_\ell \in \mathbb{Z}(p^\ell), \tag{32}$$

are additive characters in $\mathbb{Z}(p^\ell)$. $C(p^\ell)$ is isomorphic to $\mathbb{Z}(p^\ell)$.

The Prüfer p-group (or quasi-cyclic group) $C(p^\infty)$ contains all p^ℓ-th roots of unity, for all $\ell \in \mathbb{Z}^+$:

$$C(p^\infty) = \{\chi_\ell(\alpha_\ell) | \alpha_\ell \in \mathbb{Z}(p^\ell), \ell \in \mathbb{Z}^+\}. \tag{33}$$

Remark II.8. The Prüfer p-group is isomorphic to $\mathbb{Q}_p/\mathbb{Z}_p$. The correspondence between the elements of the two groups is given by:

$$\mathbb{Q}_p/\mathbb{Z}_p \ni \alpha \quad \leftrightarrow \quad \chi(\alpha) = \chi_\nu(p^\nu\alpha) \in C(p^\infty); \qquad \nu = -\mathrm{ord}(\alpha). \tag{34}$$

The following proposition is well known (e.g., p.17 in [24]):

Proposition II.9. *We consider the additive groups* $\mathbb{Z}(p^\ell)$*. For* $k \le \ell$ *we define the 'multiplication by p' homomorphisms*

$$\widetilde{\varphi}_{k\ell} : \mathbb{Z}(p^k) \rightarrow \mathbb{Z}(p^\ell); \qquad k \le \ell, \tag{35}$$

where

$$\widetilde{\varphi}_{k\ell}(\alpha_k) = \alpha_\ell; \qquad \alpha_\ell = p^{\ell-k}\alpha_k; \qquad \alpha_\ell \in \mathbb{Z}(p^\ell); \qquad \alpha_k \in \mathbb{Z}(p^k). \tag{36}$$

Then $\{\mathbb{Z}(p^\ell), \widetilde{\varphi}_{k\ell}\}$ *(for* $\ell \in \mathbb{Z}^+$*), is a direct system with direct limit*

$$\varinjlim \mathbb{Z}(p^\ell) \cong \mathbb{Q}_p/\mathbb{Z}_p \cong C(p^\infty). \tag{37}$$

Proof. If $k \le \ell \le r$, we easily see that

$$\widetilde{\varphi}_{\ell r}[\widetilde{\varphi}_{k\ell}(\alpha_k)] = \widetilde{\varphi}_{kr}(\alpha_k); \qquad k \le \ell \le r; \qquad \alpha_r \in \mathbb{Z}(p^r). \tag{38}$$

In addition to that, $\widetilde{\varphi}_{\ell\ell}$ is the identity and therefore $\{\mathbb{Z}(p^\ell), \widetilde{\varphi}_{k\ell}\}$ is a direct system.

The direct limit of this direct system is the disjoint union of all $C(p^\ell) \cong \mathbb{Z}(p^\ell)$ modulo an equivalence relation where $\alpha_\ell \sim \alpha_k$ if $\widetilde{\varphi}_{\ell r}(\alpha_\ell) = \widetilde{\varphi}_{kr}(\alpha_k)$ for some $r \ge k$ and $r \ge \ell$. Therefore $\alpha_\ell \sim \alpha_k$ if $\chi_\ell(\alpha_\ell) = \chi_k(\alpha_k)$ and the direct limit is $C(p^\infty) \cong \mathbb{Q}_p/\mathbb{Z}_p$. □

Remark II.10. There exist homomorphisms $\widetilde{\varpi}_\ell$ from $\mathbb{Z}(p^\ell)$ to $\mathbb{Q}_p/\mathbb{Z}_p$ which are given by

$$\widetilde{\varpi}_\ell(\alpha_\ell) = p^{-\ell}\alpha_\ell; \qquad \alpha_\ell \in \mathbb{Z}(p^\ell). \tag{39}$$

They are compatible in the sense that

$$k \le \ell \;\; \rightarrow \;\; \widetilde{\varpi}_\ell \circ \widetilde{\varphi}_{k\ell} = \widetilde{\varpi}_k. \tag{40}$$

Remark II.11. It is well known that $\mathbb{Q}_p/\mathbb{Z}_p$ is the Pontryagin dual group of $\mathbb{Z}_p$. It is interesting to prove it using lemma 6.4.4 in[38] (or lemma 2.9.3 in [24]). $\mathbb{Z}_p$ is the inverse limit of the surjective inverse system $\{\mathbb{Z}(p^\ell), \varphi_{k\ell}\}$. Then the Pontryagin dual groups to $\mathbb{Z}(p^\ell)$ (which are isomorphic to $\mathbb{Z}(p^\ell)$) with homomorphisms $\widetilde{\varphi}_{k\ell}$ which obey the relations

$$\widetilde{\varphi}_{k\ell}(\alpha_k) = \alpha_k \varphi_{k\ell}, \tag{41}$$

form a direct system $\{\mathbb{Z}(p^\ell), \widetilde{\varphi}_{k\ell}\}$ with direct limit the Pontryagin dual group of $\mathbb{Z}_p$. We can easily show that Eq.(41) is indeed true for the homomorphisms in Eqs(21), (36) and then using Prop.II.9 we conclude that $\mathbb{Q}_p/\mathbb{Z}_p$ is indeed the Pontryagin dual group of $\mathbb{Z}_p$.

III. HEISENBERG-WEYL GROUPS

Definition III.1. The Heisenberg-Weyl group has elements $g(a, b, c)$ that depend on three variables a, b, c which belong to a ring $\mathcal{R}$ (in our case $\mathcal{R}$ is $\mathbb{Z}(p^\ell)$ and also $\mathbb{Z}_p$) and the multiplication rule:

$$g(a_1, b_1, c_1)g(a_2, b_2, c_2) = g(a_1 + a_2, b_1 + b_2, c); \qquad c = c_1 + c_2 + 2^{-1}(a_1 b_2 - a_2 b_1). \tag{42}$$

In the rings that we consider in this paper, the inverse of 2 exists (p is an odd prime number).

Remark III.2. There are generalizations of the above definition, where b belongs to a subring $\mathcal{S}$ of $\mathcal{R}$ and a, c belong in $\mathcal{R}/\mathcal{S}$. Harmonic analysis on a circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is such an example and below we will consider the case where $\mathcal{R} = \mathbb{Q}_p$ and $\mathcal{S} = \mathbb{Z}_p$. For clarity we indicate clearly the sets in which the three parameters belong and denote the Heisenberg-Weyl group as $HW[\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3]$ where $a \in \mathcal{R}_1$, $b \in \mathcal{R}_2$, $c \in \mathcal{R}_3$.

Remark III.3. The groups

$$\begin{aligned} HW_1[\mathcal{R}_1] &= \{g(a, 0, 0) | a \in \mathcal{R}_1\}, \\ HW_2[\mathcal{R}_2] &= \{g(0, b, 0) | b \in \mathcal{R}_2\}, \\ HW_3[\mathcal{R}_3] &= \{g(0, 0, c) | c \in \mathcal{R}_3\}, \end{aligned} \tag{43}$$

are abelian subgroups of $HW[\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3]$. $HW_3[\mathcal{R}_3]$ is the centre of $HW[\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3]$ and

$$HW[\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3]/HW_3[\mathcal{R}_3] \cong HW_1[\mathcal{R}_1] \times HW_2[\mathcal{R}_2], \tag{44}$$

is an Abelian group. The $HW[\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3]$ is a central extension of the Abelian group $HW_1[\mathcal{R}_1] \times HW_2[\mathcal{R}_2]$ by the $HW_3[\mathcal{R}_3]$.

IV. THE DISCRETE HEISENBERG-WEYL GROUP $HW[\mathbb{Z}(p^\ell), \mathbb{Z}(p^\ell), \mathbb{Z}(p^\ell)]$

Definition IV.1. Let $\mathcal{H}[\mathbb{Z}(p^\ell)]$ be the p^ℓ-dimensional Hilbert space of complex vectors $f_\ell(m_\ell)$ where $m_\ell \in \mathbb{Z}(p^\ell)$. The scalar product (f_ℓ, g_ℓ) is defined as

$$(f_\ell, g_\ell) = \sum_{m_\ell \in \mathbb{Z}(p^\ell)} \overline{f_\ell(m_\ell)} g_\ell(m_\ell). \tag{45}$$

For convenience we normalize these functions so that $(f_\ell, f_\ell) = 1$.

Definition IV.2. The Fourier transform of a vector $f_\ell(m_\ell)$ in $\mathcal{H}[\mathbb{Z}(p^\ell)]$, is given by

$$\tilde{f}_\ell(n_\ell) = p^{-\ell/2} \sum_{m_\ell} f_\ell(m_\ell) \chi_\ell(n_\ell m_\ell). \tag{46}$$

Definition IV.3. The displacement operators $D_\ell(\alpha_\ell, \beta_\ell, \gamma_\ell)$ where $\alpha_\ell, \beta_\ell, \gamma_\ell \in \mathbb{Z}(p^\ell)$ are unitary $p^\ell \times p^\ell$ matrices which act on the complex vectors $f_\ell(m_\ell)$ as follows:

$$[D_\ell(\alpha_\ell, \beta_\ell, \gamma_\ell) f_\ell](m_\ell) = \chi_\ell(\gamma - 2^{-1}\alpha_\ell \beta_\ell + \alpha_\ell m_\ell) f(m_\ell - \beta_\ell). \tag{47}$$

Proposition IV.4. *The $D_\ell(\alpha_\ell, \beta_\ell, \gamma_\ell)$ form a representation of the Heisenberg-Weyl group (which we denote as $HW[\mathbb{Z}(p^\ell), \mathbb{Z}(p^\ell), \mathbb{Z}(p^\ell)]$).*

Proof. Using Eq.(47) we easily prove that the $D_\ell(\alpha_\ell, \beta_\ell, \gamma_\ell)$ form a group with the multiplication rule of Eq.(42). □

Remark IV.5. The abelian subroups defined in Eq.(43) are here

$$HW_1[\mathbb{Z}(p^\ell)] \cong HW_2[\mathbb{Z}(p^\ell)] \cong HW_3[\mathbb{Z}(p^\ell)] \cong \mathbb{Z}(p^\ell). \tag{48}$$

The $HW_1[\mathbb{Z}(p^\ell)]$ and $HW_2[\mathbb{Z}(p^\ell)]$ are Pontryagin dual to each other.

Proposition IV.6. *Let $\mathfrak{Q}_\ell(\beta_\ell)$ and $\mathfrak{P}_\ell(\alpha_\ell)$ be the 'marginal matrices'*

$$\begin{aligned}\mathfrak{Q}_\ell(2^{-1}\beta_\ell) &\equiv \frac{1}{p^\ell} \sum_{\alpha_\ell} D_\ell(\alpha_\ell, \beta_\ell, 0) \\ \mathfrak{P}_\ell(2^{-1}\alpha_\ell) &\equiv \frac{1}{p^\ell} \sum_{\beta_\ell} D_\ell(\alpha_\ell, \beta_\ell, 0).\end{aligned} \tag{49}$$

Then for any f_ℓ, g_ℓ in $\mathcal{H}[\mathbb{Z}(p^\ell)]$

$$\begin{aligned}(g_\ell, \mathfrak{Q}_\ell(\beta_\ell)\ f_\ell) &= \overline{g_\ell(2^{-1}\beta_\ell)}\ f_\ell(-2^{-1}\beta_\ell) \\ (g_\ell, \mathfrak{P}_\ell(\alpha_\ell)\ f_\ell) &= \overline{\tilde{g}_\ell(2^{-1}\alpha_\ell)}\ \tilde{f}_\ell(-2^{-1}\alpha_\ell).\end{aligned} \tag{50}$$

Proof.

$$(g_\ell, \mathfrak{Q}_\ell(\beta_\ell)\ f_\ell) = \sum_{m_\ell} \left[\overline{g_\ell(m_\ell)}\ \frac{1}{p^\ell} \sum_{\alpha_\ell} \chi_\ell(2^{-1}\alpha_\ell\beta_\ell + \alpha_\ell m_\ell)\ f_\ell(m_\ell - \beta_\ell) \right]. \tag{51}$$

The orthogonality of the characters gives

$$\frac{1}{p^\ell} \sum_{\alpha_\ell} \chi_\ell(\alpha_\ell \beta_\ell - \alpha_\ell \gamma_\ell) = \delta(\beta_\ell, \gamma_\ell), \tag{52}$$

where $\delta(\beta_\ell, \gamma_\ell)$ is Kronecker's delta. Using it we prove the first of Eqs(50). The proof of the second one is analogous to this. □

Proposition IV.7. *An arbitrary operator Θ acting on $\mathcal{H}[\mathbb{Z}(p^\ell)]$ can be expanded in terms of the displacement operators as*

$$\Theta = \frac{1}{p^\ell} \sum_{\alpha_\ell, \beta_\ell} D_\ell(\alpha_\ell, \beta_\ell, 0) \mathcal{W}(-\alpha_\ell, -\beta_\ell), \tag{53}$$

where $\mathcal{W}(\alpha, \beta)$ is the Weyl or ambiguity function

$$\mathcal{W}(\alpha_\ell, \beta_\ell) = \mathrm{tr}[D_\ell(\alpha_\ell, \beta_\ell, 0)\Theta]. \tag{54}$$

Here tr *denotes the usual trace of a matrix.*

Proof. Let $\mathcal{K}$ be the right hand side of Eq.(53). The $\mathcal{V}_{n_\ell}(m_\ell) = p^{-\ell/2}\chi_\ell(n_\ell m_\ell)$, is an orthonormal basis and it is sufficient to show that

$$\left(\mathcal{V}_{n'_\ell}, \Theta \mathcal{V}_{n_\ell}\right) = \left(\mathcal{V}_{n'_\ell}, \mathcal{K} \mathcal{V}_{n_\ell}\right). \tag{55}$$

Using Eq.(47) we prove that

$$\left(\mathcal{V}_{n'_\ell}, \mathcal{K} \mathcal{V}_{n_\ell}\right) = \frac{1}{p^{2\ell}} \sum_{\alpha_\ell, \beta_\ell, m_\ell} \mathcal{W}(-\alpha_\ell, -\beta_\ell)\ \chi_\ell(-2^{-1}\alpha_\ell\beta_\ell + \alpha_\ell m_\ell + n_\ell m_\ell - n_\ell \beta_\ell - n'_\ell m_\ell). \tag{56}$$

We then use Eq.(52) to get

$$\left(\mathcal{V}_{n'_\ell}, \mathcal{K} \mathcal{V}_{n_\ell}\right) = \frac{1}{p^\ell} \sum_{\beta_\ell} \mathcal{W}(n_\ell - n'_\ell, -\beta_\ell)\ \chi_\ell[-2^{-1}(\beta_\ell n_\ell + \beta'_\ell n'_\ell)]. \tag{57}$$

We also have

$$\begin{aligned} \mathcal{W}(n_\ell - n'_\ell, -\beta_\ell) &= \mathrm{tr}[D(n_\ell - n'_\ell, -\beta_\ell, 0)\Theta] = \sum_{k_\ell} (D(n'_\ell - n_\ell, \beta_\ell, 0)\mathcal{V}_{k_\ell}, \Theta \mathcal{V}_{k_\ell}) \\ &= \sum_{k_\ell} \left(p^{-\ell/2}\chi_\ell[(n'_\ell - n_\ell)(-2^{-1}\beta_\ell + m_\ell) + k_\ell(m_\ell - \beta_\ell)],\ \Theta \mathcal{V}_{k_\ell}(m_\ell)\right). \end{aligned} \tag{58}$$

We insert this into Eq.(57) and perform the sumation over β_ℓ taking into account Eq.(52). This proves Eq.(55). □

V. THE PROFINITE HEISENBERG-WEYL GROUP $\mathfrak{HW}[\mathbb{Z}_p, \mathbb{Z}_p, \mathbb{Z}_p]$

Lemma V.1. *We consider the $HW[\mathbb{Z}(p^\ell), \mathbb{Z}(p^\ell), \mathbb{Z}(p^\ell)]$ as multiplicative topological groups with the discrete topology.*

For $k \le \ell$ we define the homomorphisms

$$\Phi_{k\ell}: \quad HW[\mathbb{Z}(p^k), \mathbb{Z}(p^k), \mathbb{Z}(p^k)] \ \leftarrow \ HW[\mathbb{Z}(p^\ell), \mathbb{Z}(p^\ell), \mathbb{Z}(p^\ell)]; \qquad k \le \ell, \tag{59}$$

where

$$\Phi_{k\ell}[D_\ell(\alpha_\ell, \beta_\ell, \gamma_\ell)] = D_k(\alpha_k, \beta_k, \gamma_k), \tag{60}$$

and

$$\alpha_k = \varphi_{k\ell}(\alpha_\ell); \qquad \beta_k = \varphi_{k\ell}(\beta_\ell); \qquad \gamma_k = \varphi_{k\ell}(\gamma_\ell). \tag{61}$$

The map $\varphi_{k\ell}$ has been defined in Eq.(21). Then the $\{HW[\mathbb{Z}(p^\ell), \mathbb{Z}(p^\ell), \mathbb{Z}(p^\ell)], \Phi_{k\ell}\}$ is an inverse system.

Proof. If $\varphi_{k\ell}(\alpha_\ell)=\alpha_k$ and $\varphi_{k\ell}(\beta_\ell)=\beta_k$ then it is also $\varphi_{k\ell}(\alpha_\ell+\beta_\ell)=\alpha_k+\beta_k$ and $\varphi_{k\ell}(\alpha_\ell\beta_\ell)=\alpha_k\beta_k$ and $\varphi_{k\ell}(2_\ell^{-1}\beta_\ell)=2_k^{-1}\beta_k$. Here $2_k^{-1}=(p^k+1)/2$ is the inverse of 2 in $\mathbb{Z}(p^k)$. Using this and Eq.(42) for the multiplication rule of elements of the Heisenberg-Weyl group, we show that $\Phi_{k\ell}$ is a homomorphism:

$$\Phi_{k\ell}[D_\ell(\alpha_\ell,\beta_\ell,\gamma_\ell)D_\ell(\alpha_\ell',\beta_\ell',\gamma_\ell')]=\Phi_{k\ell}[D_\ell(\alpha_\ell,\beta_\ell,\gamma_\ell)]\ \Phi_{k\ell}[D_\ell(\alpha_\ell',\beta_\ell',\gamma_\ell')]. \quad (62)$$

$\Phi_{k\ell}$ are continuous maps (because their domains have the discrete topology) and for $k\le\ell\le r$

$$\Phi_{k\ell}\{\Phi_{\ell r}[D_r(\alpha_r,\beta_r,\gamma_r)]\}=\Phi_{kr}[D_r(\alpha_r,\beta_r,\gamma_r)]. \quad (63)$$

The Φ_{kk} is the identity and therefore the $\{HW[\mathbb{Z}(p^\ell),\mathbb{Z}(p^\ell),\mathbb{Z}(p^\ell)],\Phi_{k\ell}\}$ is an inverse system. □

Theorem V.2.

(1) The inverse limit of the inverse system $\{HW[\mathbb{Z}(p^\ell),\mathbb{Z}(p^\ell),\mathbb{Z}(p^\ell)],\Phi_{k\ell}\}$ is a representation of the Heisenberg-Weyl group over $\mathbb{Z}_p$ which we denote as $\mathfrak{HW}[\mathbb{Z}_p,\mathbb{Z}_p,\mathbb{Z}_p]$

(2) The $\mathfrak{HW}[\mathbb{Z}_p,\mathbb{Z}_p,\mathbb{Z}_p]$ is a profinite group

Proof. The elements of the inverse limit of the inverse system $\{HW[\mathbb{Z}(p^\ell),\mathbb{Z}(p^\ell),\mathbb{Z}(p^\ell)],\Phi_{k\ell}\}$ are the sequences

$$\begin{aligned}&\mathfrak{D}(a,b,c)\equiv\{D_1(\alpha_1,\beta_1,\gamma_1),D_2(\alpha_2,\beta_2,\gamma_2),...\};\qquad \alpha_\ell,\beta_\ell,\gamma_\ell\in\mathbb{Z}(p^\ell);\qquad a,b,c\in\mathbb{Z}_p\\&\alpha_\ell=\varpi_\ell(a);\qquad \beta_\ell=\varpi_\ell(b);\qquad \gamma_\ell=\varpi_\ell(c),\end{aligned} \quad (64)$$

where ϖ_ℓ are the projections in Eq.(28). Multiplication is performed componentwise and we easily see that it satisfies Eq.(42). Therefore we have a representation of the Heisenberg-Weyl group.

The fact that $\mathfrak{HW}[\mathbb{Z}_p,\mathbb{Z}_p,\mathbb{Z}_p]$ is an inverse limit, proves that it is a profinite group.

□

Remark V.3 (projections). The natural projection homomorphisms from $\mathfrak{HW}[\mathbb{Z}_p,\mathbb{Z}_p,\mathbb{Z}_p]$ to $HW[\mathbb{Z}(p^r),\mathbb{Z}(p^r),\mathbb{Z}(p^r)]$ are:

$$\Pi_r[\mathfrak{D}(a,b,c)]=D_r(\alpha_r,\beta_r,\gamma_r). \quad (65)$$

These projections are compatible:

$$k\le r\quad\rightarrow\quad \Phi_{kr}\circ\Pi_r=\Pi_k. \quad (66)$$

Remark V.4 (topology). As a profinite group the $\mathfrak{HW}[\mathbb{Z}_p,\mathbb{Z}_p,\mathbb{Z}_p]$ is a Hausdorff compact totally disconnected topological group and below we give a fundamental system of neighbourhoods of the identity.

Definition V.5. The subgroup $\mathfrak{HW}[p^{\ell-1}\mathbb{Z}_p,p^{\ell-1}\mathbb{Z}_p,p^{\ell-1}\mathbb{Z}_p]$ of $\mathfrak{HW}[\mathbb{Z}_p,\mathbb{Z}_p,\mathbb{Z}_p]$ has as elements the sequences

$$\begin{aligned}&\mathfrak{D}(a,b,c)=\{\mathbf{1},...,\mathbf{1},D_\ell(\alpha_\ell,\beta_\ell,\gamma_\ell),D_{\ell+1}(\alpha_{\ell+1},\beta_{\ell+1},\gamma_{\ell+1}),...\}\\&\alpha_k=\varpi_k(a);\qquad \beta_k=\varpi_k(b);\qquad \gamma_k=\varpi_k(c);\qquad \alpha_k,\beta_k,\gamma_k\in\mathbb{Z}(p^k);\qquad a,b,c\in p^{\ell-1}\mathbb{Z}_p,\end{aligned} \quad (67)$$

where ϖ_k are the projections in Eq.(28). In other words

$$\alpha_\ell\ =\ \hat\alpha_{\ell-1}p^{\ell-1};\qquad \alpha_{\ell+1}=\hat\alpha_{\ell-1}p^{\ell-1}+\hat\alpha_\ell p^\ell;\quad ..., \quad (68)$$

and similarly for the β_ℓ,γ_ℓ.

Proposition V.6. *The*

$$\mathfrak{HW}[\mathbb{Z}_p,\mathbb{Z}_p,\mathbb{Z}_p]\supset\mathfrak{HW}[p\mathbb{Z}_p,p\mathbb{Z}_p,p\mathbb{Z}_p]\supset\mathfrak{HW}[p^2\mathbb{Z}_p,p^2\mathbb{Z}_p,p^2\mathbb{Z}_p]\supset..., \quad (69)$$

is a fundamental system of neighbourhoods of the identity of $\mathfrak{HW}[\mathbb{Z}_p,\mathbb{Z}_p,\mathbb{Z}_p]$.

Proof. This is consequence of the fundamental system of neighbourhoods of 0 in $\mathbb{Z}_p$ discussed in remark II.6. □

Remark V.7. The abelian subgroups defined in Eq.(43) are here:

$$\mathfrak{HW}_1[\mathbb{Z}_p] \cong \mathfrak{HW}_2[\mathbb{Z}_p] \cong \mathfrak{HW}_3[\mathbb{Z}_p] \cong \mathbb{Z}_p. \tag{70}$$

The $\mathfrak{HW}_1[\mathbb{Z}_p]$ and $\mathfrak{HW}_2[\mathbb{Z}_p]$ are both isomorphic to $\mathbb{Z}_p$ and therefore they are **not** Pontryagin dual to each other. In this sense, this section is not relevant to harmonic analysis. It is a contribution to the representations of the Heisenberg-Weyl group.

Remark V.8 (Direct limit). In this section we have considered an inverse system that involves the $HW[\mathbb{Z}(p^\ell), \mathbb{Z}(p^\ell), \mathbb{Z}(p^\ell)]$. It is natural to consider the question whether there is a direct system that involves the $HW[\mathbb{Z}(p^\ell), \mathbb{Z}(p^\ell), \mathbb{Z}(p^\ell)]$.

For $k \le \ell$ we define the maps

$$S_{k\ell}: \;\; HW[\mathbb{Z}(p^k), \mathbb{Z}(p^k), \mathbb{Z}(p^k)] \;\rightarrow\; HW[\mathbb{Z}(p^\ell), \mathbb{Z}(p^\ell), \mathbb{Z}(p^\ell)]; \qquad k \le \ell, \tag{71}$$

where

$$S_{k\ell}[D_k(\alpha_k, \beta_k, \gamma_k)] = D_\ell(\alpha_\ell, \beta_\ell, \gamma_\ell), \tag{72}$$

and

$$\alpha_\ell = \widetilde{\varphi}_{k\ell}(\alpha_k); \qquad \beta_\ell = \widetilde{\varphi}_{k\ell}(\beta_k); \qquad \gamma_\ell = \widetilde{\varphi}_{k\ell}(\gamma_k). \tag{73}$$

The map $\widetilde{\varphi}_{k\ell}$ has been defined in Eq.(36). In this case the $S_{k\ell}[D_k(\alpha_k, \beta_k, \gamma_k)\; D_k(\alpha'_k, \beta'_k, \gamma'_k)]$ is not equal to $S_{k\ell}[D_k(\alpha_k, \beta_k, \gamma_k)]S_{k\ell}[D_k(\alpha'_k, \beta'_k, \gamma'_k)]$. This is due to the fact that $\widetilde{\varphi}_{k\ell}(\alpha_k\beta_k)$ is not equal to $\widetilde{\varphi}_{k\ell}(\alpha_k)\widetilde{\varphi}_{k\ell}(\beta_k)$. Therefore the $S_{k\ell}$ are not homomorphisms and the $\{HW[\mathbb{Z}(p^\ell), \mathbb{Z}(p^\ell), \mathbb{Z}(p^\ell)], S_{k\ell}\}$ is **not** a direct system.

Of course, if we restrict the domain of $S_{k\ell}$ into one of the subgroups $HW_1[\mathbb{Z}(p^\ell)]$ or $HW_2[\mathbb{Z}(p^\ell)]$ or $HW_3[\mathbb{Z}(p^\ell)]$ then it is homomorphism. The corresponding direct systems have direct limits isomorphic to $\mathbb{Q}_p/\mathbb{Z}_p$.

VI. HARMONIC ANALYSIS ON $\mathbb{Z}_p$

We consider complex functions $f(x)$, where $x \in \mathbb{Z}_p$. These functions have degree of compact support $k \ge 0$ and in addition to that they are taken to be locally constant.

The Fourier transform is given by

$$\tilde{f}(\mathfrak{p}) = \int_{\mathbb{Z}_p} dx\; \chi(-x\mathfrak{p}) f(x); \qquad \mathfrak{p} \in \mathbb{Q}_p/\mathbb{Z}_p. \tag{74}$$

The inverse Fourier transform is given by

$$f(x) = \int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{p}\; \chi(x\mathfrak{p})\tilde{f}(\mathfrak{p}). \tag{75}$$

Proposition VI.1. *If a function $f(x)$ (where $x \in \mathbb{Z}_p$) has compact support with degree k and is locally constant with degree n, then its Fourier transform $\tilde{f}(\mathfrak{p})$ (where $\mathfrak{p} \in \mathbb{Q}_p/\mathbb{Z}_p$) has compact support with degree n and is locally constant with degree k.*

Proof. If the function $f(x)$ is locally constant with degree n, it obeys Eq.(11). Then for $|\alpha| \le p^{-n}$ we show that

$$\int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{p}\; \chi(x\mathfrak{p})\tilde{f}(\mathfrak{p})[1 - \chi(\alpha\mathfrak{p})] = 0. \tag{76}$$

It is seen that the Fourier transform of $\tilde{f}(\mathfrak{p})[1 - \chi(\alpha\mathfrak{p})]$ is zero and consequently $\tilde{f}(\mathfrak{p})[1 - \chi(\alpha\mathfrak{p})] = 0$. For $|\alpha| \le p^{-n}$ and $|\mathfrak{p}| > p^n$ the $1 - \chi(\alpha\mathfrak{p}) \ne 0$ and therefore $\tilde{f}(\mathfrak{p}) = 0$. Therefore the function $\tilde{f}(\mathfrak{p})$ has compact support with degree n.

The second part of the proposition is proved in a similar way. □

Definition VI.2.

(1) $\mathbf{H}[\mathbb{Z}_p]$ is the inner product space of complex locally constant functions $f(x)$ $(x \in \mathbb{Z}_p)$ with the inner product

$$(f,g) = \int_{\mathbb{Z}_p} dx\ \overline{f(x)}\ g(x). \tag{77}$$

These functions have compact support with degree $k \geq 0$.

(2) $\mathbf{H}[\mathbb{Q}_p/\mathbb{Z}_p]$ is the inner product space of complex functions $F(\mathfrak{p})$ $(\mathfrak{p} \in \mathbb{Q}_p/\mathbb{Z}_p)$ with compact support, with the inner product

$$(F,G) = \int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{p}\ \overline{F(\mathfrak{p})}\ G(\mathfrak{p}). \tag{78}$$

These functions are locally constant with degree $k \geq 0$.

Remark VI.3 (completeness). Following ref.[10] we introduce a topology in $\mathbf{H}[\mathbb{Z}_p]$ as follows: We say that a sequence of functions $f_N(x)$ in $\mathbf{H}[\mathbb{Z}_p]$ tends to zero if there exists an integer K such that all functions $f_N(x)$ are locally constant with degree less or equal to K and in addition to that for $N \to \infty$ the $f_N(x)$ go to zero, uniformly.

Remark VI.4. The general theory of harmonic analysis on locally compact Abelian groups proves that $\mathbf{H}[\mathbb{Z}_p]$ and $\mathbf{H}[\mathbb{Q}_p/\mathbb{Z}_p]$ are isomorphic to each other (and in the following we use the same notation $\mathbf{H}$ for both). It also proves that Parceval's relation $(\tilde{f}, \tilde{g}) = (f,g)$, is valid.

Generalized functions (delta functions) which are outside the space $\mathbf{H}$, are sometimes needed in practical applications and they have been discussed in refs[9, 10, 34]

A. The totally disconnected and locally compact Heisenberg-Weyl group $\mathsf{HW}[(\mathbb{Q}_p/\mathbb{Z}_p), \mathbb{Z}_p, (\mathbb{Q}_p/\mathbb{Z}_p)]$

Definition VI.5. The displacement operators $D(\mathfrak{a}, b, \mathfrak{c})$ where $\mathfrak{a}, \mathfrak{c} \in \mathbb{Q}_p/\mathbb{Z}_p$ and $b \in \mathbb{Z}_p$ act on the functions $f(x)$ as follows:

$$[D(\mathfrak{a}, b, \mathfrak{c})f](x) = \chi\left(\mathfrak{c} - \frac{1}{2}\mathfrak{a}b + \mathfrak{a}x\right) f(x-b). \tag{79}$$

Theorem VI.6. *The $D(\mathfrak{a}, b, \mathfrak{c})$ form a representation of the Heisenberg-Weyl group, which we denote as* $\mathsf{HW}[(\mathbb{Q}_p/\mathbb{Z}_p), \mathbb{Z}_p, (\mathbb{Q}_p/\mathbb{Z}_p)]$.

Proof. Using Eq.(79) we easily prove that the $D(\mathfrak{a}, b, \mathfrak{c})$ form a group with the multiplication rule of Eq.(42). □

We consider the abelian subgroups $\mathsf{HW}_1[\mathbb{Q}_p/\mathbb{Z}_p]$, $\mathsf{HW}_2[\mathbb{Z}_p]$ and $\mathsf{HW}_3[\mathbb{Q}_p/\mathbb{Z}_p]$ of $\mathsf{HW}[(\mathbb{Q}_p/\mathbb{Z}_p), \mathbb{Z}_p, (\mathbb{Q}_p/\mathbb{Z}_p]$ defined in Eq.(43) and show easily the isomorphisms:

$$\mathsf{HW}_1[\mathbb{Q}_p/\mathbb{Z}_p] \cong \mathsf{HW}_3[\mathbb{Q}_p/\mathbb{Z}_p] \cong \mathbb{Q}_p/\mathbb{Z}_p; \qquad \mathsf{HW}_2[\mathbb{Z}_p] \cong \mathbb{Z}_p. \tag{80}$$

We also consider the subgroup with elements $D(0, b, 0)$ where $b \in p^n\mathbb{Z}_p$, which we denote as $\mathsf{HW}_2[p^n\mathbb{Z}_p]$. Clearly

$$\mathsf{HW}[(\mathbb{Q}_p/\mathbb{Z}_p), \mathbb{Z}_p, (\mathbb{Q}_p/\mathbb{Z}_p)] \supset \mathsf{HW}_2[\mathbb{Z}_p] \supset \mathsf{HW}_2[p\mathbb{Z}_p] \supset \mathsf{HW}_2[p^2\mathbb{Z}_p] \supset ... \tag{81}$$

Notation VI.7. *Let A, B be subsets of a group G and $g \in G$. Then*

$$gA = \{ga | a \in A\}; \qquad gAg^{-1} = \{gag^{-1} | a \in A\}; \qquad AB = \bigcup_{a \in A} aB; \qquad A^{-1} = \{a^{-1} | a \in A\}. \tag{82}$$

Lemma VI.8. $\mathsf{HW}[(\mathbb{Q}_p/\mathbb{Z}_p), \mathbb{Z}_p, (\mathbb{Q}_p/\mathbb{Z}_p)]$ *can be made into a topological group by specifying as a fundamental system of open neighbourhoods of the identity the subgroups in Eq.(81).*

Proof. Let $\mathfrak{N}$ be the set of the subgroups of Eq.(81). We need to prove that they do satisfy the properties of a fundamental system of open neighbourhoods of the identity (section III.1.2 in[3])

The first property is that given any $U \in \mathfrak{N}$ there exists $V \in \mathfrak{N}$ such that $VV \subset U$. Indeed for $U = \mathsf{HW}_2[p^n\mathbb{Z}_p]$ all the $V = \mathsf{HW}_2[p^k\mathbb{Z}_p]$ with $k \geq n$ satisfy this.

The second property is that given any $U \in \mathfrak{N}$ there exists $V \in \mathfrak{N}$ such that $V^{-1} \subset U$. This is true because for $U = \mathsf{HW}_2[p^n\mathbb{Z}_p]$ all the $V^{-1} = V = \mathsf{HW}_2[p^k\mathbb{Z}_p]$ with $k \geq n$ satisfy this.

The third property is that given any element $D(\mathfrak{a}, b, \mathfrak{c})$ of $\mathsf{HW}[(\mathbb{Q}_p/\mathbb{Z}_p), \mathbb{Z}_p, (\mathbb{Q}_p/\mathbb{Z}_p)]$ and any $U \in \mathfrak{N}$, there exists $V \in \mathfrak{N}$ such that $V \subset D(\mathfrak{a}, b, \mathfrak{c})UD(-\mathfrak{a}, -b, -\mathfrak{c})$. Indeed

$$D(\mathfrak{a}, b, \mathfrak{c})\ D(0, b', 0)\ D(-\mathfrak{a}, -b, -\mathfrak{c}) = D(0, b', \mathfrak{a}b'). \tag{83}$$

Therefore for $U = \mathsf{HW}_2[p^n\mathbb{Z}_p]$ all the $V = \mathsf{HW}_2[p^k\mathbb{Z}_p]$ with $k \geq \max(n, -\mathrm{ord}(\mathfrak{a}))$ satisfy this (in this case $\mathfrak{a}b'$ is the zero element of $\mathbb{Q}_p/\mathbb{Z}_p$).

Therefore there is a unique topology on $\mathsf{HW}[(\mathbb{Q}_p/\mathbb{Z}_p), \mathbb{Z}_p, (\mathbb{Q}_p/\mathbb{Z}_p)]$ compatible with its group structure [3]. □

Theorem VI.9. *The topological group* $\mathsf{HW}[(\mathbb{Q}_p/\mathbb{Z}_p), \mathbb{Z}_p, (\mathbb{Q}_p/\mathbb{Z}_p)]$ *is totally disconnected and locally compact.*

Proof. Topologically the $\mathsf{HW}_2[\mathbb{Z}_p] \cong \mathbb{Z}_p$ is compact and totally disconnected (therefore it is also locally compact and totally disconnected); and the $\mathsf{HW}_1[\mathbb{Q}_p/\mathbb{Z}_p] \cong \mathsf{HW}_3[\mathbb{Q}_p/\mathbb{Z}_p] \cong \mathbb{Q}_p/\mathbb{Z}_p$ are discrete (and therefore locally compact and totally disconnected).

Since both $\mathsf{HW}_1[\mathbb{Q}_p/\mathbb{Z}_p]$ and $\mathsf{HW}_2[\mathbb{Z}_p]$, are locally compact and totally disconnected, the $\mathsf{HW}_1[\mathbb{Q}_p/\mathbb{Z}_p] \times \mathsf{HW}_2[\mathbb{Z}_p]$ with the product topology, is also locally compact and totally disconnected. Eq.(44) in the present context is

$$\mathsf{HW}[(\mathbb{Q}_p/\mathbb{Z}_p), \mathbb{Z}_p, (\mathbb{Q}_p/\mathbb{Z}_p)]/\mathsf{HW}_3[\mathbb{Q}_p/\mathbb{Z}_p] \cong \mathsf{HW}_1[\mathbb{Q}_p/\mathbb{Z}_p] \times \mathsf{HW}_2[\mathbb{Z}_p]. \tag{84}$$

It is seen that both $\mathsf{HW}[(\mathbb{Q}_p/\mathbb{Z}_p), \mathbb{Z}_p, (\mathbb{Q}_p/\mathbb{Z}_p)]/\mathsf{HW}_3[\mathbb{Q}_p/\mathbb{Z}_p]$ and also $\mathsf{HW}_3[\mathbb{Q}_p/\mathbb{Z}_p]$ are locally compact and totally disconnected and therefore (e.g.[3]) the $\mathsf{HW}[(\mathbb{Q}_p/\mathbb{Z}_p), \mathbb{Z}_p, (\mathbb{Q}_p/\mathbb{Z}_p)]$ is also locally compact and totally disconnected. □

Remark VI.10. In contrast to the case in the previous section, here the $\mathsf{HW}_1[\mathbb{Q}_p/\mathbb{Z}_p] \cong \mathbb{Q}_p/\mathbb{Z}_p$ and $\mathsf{HW}_2[\mathbb{Z}_p] \cong \mathbb{Z}_p$ are Pontryagin dual to each other.

Theorem VI.11. *Let* $\mathfrak{Q}(b)$ *and* $\mathfrak{P}(\mathfrak{a})$ *be the 'marginal operators'*

$$\mathfrak{Q}(b) \equiv \int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{a}\ D(\mathfrak{a}, b, 0); \qquad \mathfrak{P}(\mathfrak{a}) \equiv \int_{\mathbb{Z}_p} db\ D(\mathfrak{a}, b, 0). \tag{85}$$

Then for any $g(x), f(x)$ *in* $\mathbf{H}[\mathbb{Z}_p]$

$$(g, \mathfrak{Q}(b)f) = \overline{g\,(2^{-1}b)}\ f(-2^{-1}b); \qquad (g, \mathfrak{P}(\mathfrak{a})f) = \overline{\tilde{g}(2^{-1}\mathfrak{a})}\ \tilde{f}(-2^{-1}\mathfrak{a}). \tag{86}$$

Proof. We first use Eq.(79) to prove that

$$(\mathcal{U}_{\mathfrak{p}'}, D(\mathfrak{a}, b, 0)\mathcal{U}_{\mathfrak{p}}) = \chi\left(-\frac{1}{2}\mathfrak{a}b - \mathfrak{p}b\right)\Delta(\mathfrak{p} - \mathfrak{p}' + \mathfrak{a}), \tag{87}$$

where $\mathcal{U}_{\mathfrak{p}}(x) \equiv \chi(x\mathfrak{p})$. Then we integrate and get

$$\begin{aligned}\int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{a}\ \ (\mathcal{U}_{\mathfrak{p}'}, D(\mathfrak{a}, b, 0)\mathcal{U}_{\mathfrak{p}}) &= \chi\left[-\frac{b}{2}(\mathfrak{p} + \mathfrak{p}')\right] \\ \int_{\mathbb{Z}_p} db\ \ (\mathcal{U}_{\mathfrak{p}'}, D(\mathfrak{a}, b, 0)\mathcal{U}_{\mathfrak{p}}) &= \Delta(\mathfrak{p} - \mathfrak{p}' + \mathfrak{a})\Delta\left(\mathfrak{p} + \frac{1}{2}\mathfrak{a}\right).\end{aligned} \tag{88}$$

Using them in conjuction with Eq.(75) proves the theorem. □

Definition VI.12. Let Θ be a trace class operator acting on the functions $f(x)$ $(x \in \mathbb{Z}_p)$ through a kernel $\Theta(x, x')$ as follows:

$$[\Theta f](x) = \int_{\mathbb{Z}_p} dx\ \Theta(x, x') f(x'). \tag{89}$$

The trace of this operator is given by:

$$\mathrm{tr}\Theta = \int_{\mathbb{Z}_p} dx\ \Theta(x, x). \tag{90}$$

Theorem VI.13. *Θ can be expanded in terms of displacement operators, as*

$$\Theta = \int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{a} \int_{\mathbb{Z}_p} db\ D(\mathfrak{a}, b, 0)\mathcal{W}(-\mathfrak{a}, -b), \tag{91}$$

where $\mathcal{W}(\mathfrak{a}, b)$ is the Weyl or ambiguity function:

$$\mathcal{W}(\mathfrak{a}, b) = \mathrm{tr}[\Theta D(\mathfrak{a}, b, 0)] \tag{92}$$

Proof. Let $\mathcal{L}$ be the right hand side of Eq.(91). Since $\mathcal{U}_{\mathfrak{p}}(x)$ is an orthonormal basis it is sufficient to show that

$$(\mathcal{U}_{\mathfrak{p}'}, \Theta\mathcal{U}_{\mathfrak{p}}) = (\mathcal{U}_{\mathfrak{p}'}, \mathcal{L}\mathcal{U}_{\mathfrak{p}})\,. \tag{93}$$

Using Eq.(87) we prove that

$$(\mathcal{U}_{\mathfrak{p}'}, \mathcal{L}\mathcal{U}_{\mathfrak{p}}) = \int_{\mathbb{Z}_p} db\ \mathcal{W}(\mathfrak{p} - \mathfrak{p}', -b)\ \chi\left[-\frac{1}{2}b(\mathfrak{p} + \mathfrak{p}')\right]. \tag{94}$$

We also have

$$\mathcal{W}(\mathfrak{a}, b) = \mathrm{tr}[D(\mathfrak{a}, b, 0)\Theta] = \int_{\mathbb{Z}_p} dx\ \chi\left(-\frac{1}{2}\mathfrak{a}b + \mathfrak{a}x\right)\Theta(x - b, x). \tag{95}$$

Inserting Eq.(95) into Eq.(94) using $x' = x + b$ we get

$$(\mathcal{U}_{\mathfrak{p}'}, \mathcal{L}\mathcal{U}_{\mathfrak{p}}) = \int_{\mathbb{Z}_p} dx' \int_{\mathbb{Z}_p} dx\ \chi(x\mathfrak{p} - x'\mathfrak{p}')\ \Theta(x', x) = (\mathcal{U}_{\mathfrak{p}'}, \Theta\mathcal{U}_{\mathfrak{p}})\,. \tag{96}$$

□

B. Harmonic analysis on $\mathbb{Z}_p/p^\ell\mathbb{Z}_p$

This subsection links harmonic analysis on $\mathbb{Z}_p$ discussed in this section, with harmonic analysis on $\mathbb{Z}(p^\ell)$ discussed earlier in section IV.

Definition VI.14. $\mathbf{H}[\mathbb{Z}_p/p^\ell\mathbb{Z}_p]$ is the p^ℓ-dimensional subspace of $\mathbf{H}$ which contains functions $f(x)$ where $x \in \mathbb{Z}_p/p^\ell\mathbb{Z}_p \cong \mathbb{Z}(p^\ell)$. The scalar product of Eq.(77) reduces in this case to

$$(f, g) = p^{-\ell} \sum_{x \in \mathbb{Z}_p/p^\ell\mathbb{Z}_p} \overline{f(x)}\ g(x). \tag{97}$$

Both $\mathbf{H}[\mathbb{Z}_p/p^\ell\mathbb{Z}_p]$ and $\mathcal{H}[\mathbb{Z}(p^\ell)]$ are p^ℓ-dimensional and therefore are isomorphic to each other.

Definition VI.15. Let $\mathfrak{R}_\ell$ be the following linear map from $\mathcal{H}[\mathbb{Z}(p^\ell)]$ to $\mathbf{H}[\mathbb{Z}_p/p^\ell\mathbb{Z}_p]$:

$$\mathfrak{R}_\ell[f_\ell(m_\ell)] = p^{-\ell/2}f(x); \qquad x = m_\ell \in \mathbb{Z}_p/p^\ell\mathbb{Z}_p \cong \mathbb{Z}(p^\ell). \tag{98}$$

Remark VI.16. The normalization prefactor $p^{-\ell/2}$ is used so that the scalar product (f,g) in $\mathbf{H}[\mathbb{Z}_p/p^\ell\mathbb{Z}_p]$ given in Eq.(97), is equal to the scalar product of the corresponding functions (f_ℓ, g_ℓ) in $\mathcal{H}[\mathbb{Z}(p^\ell)]$ given in Eq.(45).

Remark VI.17. A consequence of Prop.VI.1 is that the Fourier transform of $f(x)$ where $x \in \mathbb{Z}_p/p^\ell\mathbb{Z}_p$, is a function $\tilde{f}(\mathfrak{p})$ where $\mathfrak{p} \in p^{-\ell}\mathbb{Z}_p/\mathbb{Z}_p$. Then it is easily seen that

$$\mathfrak{R}_\ell[\tilde{f}_\ell(m_\ell)] = \tilde{f}(\mathfrak{p}); \qquad \mathfrak{p} = p^{-\ell}m_\ell. \tag{99}$$

Definition VI.18. Let $\mathsf{HW}[p^{-\ell}\mathbb{Z}_p/\mathbb{Z}_p, \mathbb{Z}_p/p^\ell\mathbb{Z}_p, p^{-\ell}\mathbb{Z}_p/\mathbb{Z}_p]$ be the subgroup of $\mathsf{HW}[(\mathbb{Q}_p/\mathbb{Z}_p), \mathbb{Z}_p, (\mathbb{Q}_p/\mathbb{Z}_p)]$, that consists of the displacement operators acting on the functions in $\mathbf{H}[\mathbb{Z}_p/p^\ell\mathbb{Z}_p]$. The $HW[\mathbb{Z}(p^\ell), \mathbb{Z}(p^\ell), \mathbb{Z}(p^\ell)]$ is isomorphic to $\mathsf{HW}[p^{-\ell}\mathbb{Z}_p/\mathbb{Z}_p, \mathbb{Z}_p/p^\ell\mathbb{Z}_p, p^{-\ell}\mathbb{Z}_p/\mathbb{Z}_p]$ and let $\mathfrak{T}_\ell$ be the following map between the elements of these two groups:

$$\begin{aligned}&\mathfrak{T}_\ell[D_\ell(\alpha_\ell, \beta_\ell, \gamma_\ell)] = D(\mathfrak{a}, b, \mathfrak{c})\\ &\mathfrak{a} = \widetilde{\varpi}_\ell(\alpha_\ell) = p^{-\ell}\alpha_\ell; \qquad b = \beta_\ell; \qquad \mathfrak{c} = \widetilde{\varpi}_\ell(\gamma_\ell) = p^{-\ell}\gamma_\ell,\end{aligned} \tag{100}$$

where $\mathfrak{a}, \mathfrak{c} \in p^{-\ell}\mathbb{Z}_p/\mathbb{Z}_p \cong \mathbb{Z}(p^\ell)$ and $b \in \mathbb{Z}_p/p^\ell\mathbb{Z}_p \cong \mathbb{Z}(p^\ell)$. The $\widetilde{\varpi}_\ell$ have been defined in Eq.(39).

Proposition VI.19.

$$\mathfrak{R}_\ell[D_\ell(\alpha_\ell, \beta_\ell, \gamma_\ell) f_\ell(m_\ell)] = \mathfrak{T}_\ell[D_\ell(\alpha_\ell, \beta_\ell, \gamma_\ell)]\, \mathfrak{R}_\ell[f_\ell(m_\ell)]. \tag{101}$$

Proof. The proof is based on Eqs(47), (79). □

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