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Wigner and Weyl functions for p-adic quantum mechanics

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A quantum system with positions in $\mathbb{Z}_p$ and momenta in $\mathbb{Q}_p/\mathbb{Z}_p$ is studied. The displacement operators and also the displaced parity operators in the $\mathbb{Z}_p \times \mathbb{Q}_p/\mathbb{Z}_p$ phase space of this system, are studied. The Weyl functions (which are intimately related to the displacement operators) and the Wigner functions (which are intimately related to the displaced parity operators) are discussed.

I. INTRODUCTION

Quantum mechanics and quantum field theory in the field $\mathbb{Q}_p$ of p-adic numbers have been studied extensively in the literature[1–5]. Mathematical literature relevant to these problems can be found in [6]. Related work on wavelets on locally compact fields has been presented in [7–9]. Quantum mechanics on adeles has been studied in [10]. Both cases of complex wavefunctions (of p-adic variables) and also p-adic valued wavefunctions (of p-adic variables) have been studied in the literature and here we are interested in the case of complex wavefunctions of p-adic variables.

In a recent paper [11] we have studied a system where the position takes values in the ring $\mathbb{Z}_p$ of p-adic integers and consequently the momentum takes values in $\mathbb{Q}_p/\mathbb{Z}_p$. In the present paper we review some of this work and extend it in the direction of Wigner and Weyl functions.

In section II we discuss some aspects of the p-adic formalism which are needed later. In section III we present very briefly quantum mechanics for systems with positions in $\mathbb{Z}_p$ and momenta in $\mathbb{Q}_p/\mathbb{Z}_p$ (see also [11]). In section IV we discuss the displacement operators and the corresponding Heisenberg-Weyl group. In section V we study the displaced parity operators and their properties.

The Weyl functions are intimately related to the displacement operators and the Wigner functions are intimately related to the displaced parity operators. The properties of the displacement operators and the displaced parity operators lead naturally to corresponding properties for the Wigner and Weyl functions which are presented in section VI. We conclude in section VII with a discussion of our results.

II. PRELIMINARIES

A p-adic number ($\alpha \in \mathbb{Q}_p$) can be represented as a series

$$\alpha = \sum_{\nu=\mathrm{ord}(\alpha)}^{\infty} \overline{\alpha}_\nu p^\nu; \qquad 0 \le \overline{\alpha}_\nu \le p-1 \tag{1}$$

The absolute value is given by $|\alpha| = p^{-\mathrm{ord}(\alpha)}$.

A p-adic integer ($x \in \mathbb{Z}_p$) has order greater or equal to zero. A p-adic number modulo p-adic integers

($\mathfrak{p}$ in $\mathbb{Q}_p/\mathbb{Z}_p$) is a coset and we represent it with the element which has integer part equal to zero:

$$\mathfrak{p} = \overline{\mathfrak{p}}_{-k}p^{-k} + \overline{\mathfrak{p}}_{-k+1}p^{-k+1} + ... + \overline{\mathfrak{p}}_{-1}p^{-1}; \qquad 0 \leq \overline{\mathfrak{p}}_i \leq p-1; \qquad k = -\mathrm{ord}(\mathfrak{p}) \tag{2}$$

The product $x\mathfrak{p}$ where $x \in \mathbb{Z}_p$ and $\mathfrak{p} \in \mathbb{Q}_p/\mathbb{Z}_p$ is also a coset and we represent it with the element which has integer part equal to zero.

As additive topological groups $\mathbb{Z}_p$ is a profinite group [12] and $\mathbb{Q}_p/\mathbb{Z}_p$ is a discrete group which is Pontryagin dual to $\mathbb{Z}_p$. In the present context, $x \in \mathbb{Z}_p$ are positions and $\mathfrak{p} \in \mathbb{Q}_p/\mathbb{Z}_p$ are momenta.

A complex function $f(x)$ (where $x \in \mathbb{Q}_p$) is locally constant with degree n if $f(x+\alpha) = f(x)$ for $|\alpha| \leq p^{-n}$ and has compact support with degree k, if $f(x) = 0$ for $|x| > p^k$. It has been proved in [11] that if a function $f(x)$ is locally constant with degree n and has compact support with degree k then its Fourier transform $\tilde{f}(\mathfrak{p})$ (defined below) is locally constant with degree k and has compact support with degree n. This can be interpreted as a form of the uncertainty principle in the present context (large momenta correspond to small distances and vice-versa).

In our case we have complex functions $f(x)$ where $x \in \mathbb{Z}_p$. Therefore these functions have compact support with degree $k \geq 0$ and this implies that their Fourier transforms $\tilde{f}(\mathfrak{p})$ are locally constant with degree k. In addition to that we assume that the functions $f(x)$ are locally constant with some degree n and this implies that their Fourier transforms $\tilde{f}(\mathfrak{p})$ have compact support with degree n. Integrals of $f(x)$ over $\mathbb{Z}_p$ (with the Haar measure normalized as $\int_{\mathbb{Z}_p} dx = 1$) are given by:

$$\int_{\mathbb{Z}_p} f(x)dx = p^{-n} \sum f(\overline{x}_0 + \overline{x}_1 p + ... + \overline{x}_{n-1}p^{n-1}) \tag{3}$$

Integrals of $g(\mathfrak{p})$ over $\mathbb{Q}_p/p^n\mathbb{Z}_p$ are given by

$$\int_{\mathbb{Q}_p/p^n\mathbb{Z}_p} g(\mathfrak{p})d\mathfrak{p} = p^{-n} \sum g(\overline{\mathfrak{p}}_{-k}p^{-k} + \overline{\mathfrak{p}}_{-k+1}p^{-k+1} + ... + \overline{\mathfrak{p}}_{n-1}p^{n-1}) \tag{4}$$

Additive characters are defined as follows. If $\alpha = \sum \overline{\alpha}_\nu p^\nu$ then

$$\begin{aligned} \chi(\alpha) &= \exp\left(i2\pi \sum_{\nu=\mathrm{ord}(\alpha)}^{-1} \alpha_\nu p^\nu\right); \qquad \mathrm{ord}(\alpha) \leq -1 \\ \chi(\alpha) &= 1; \qquad \mathrm{ord}(\alpha) \geq 0. \end{aligned} \tag{5}$$

It can be shown that

$$\int_{\mathbb{Z}_p} \chi(x\mathfrak{p})dx = \Delta(\mathfrak{p}); \qquad \int_{\mathbb{Q}_p/\mathbb{Z}_p} \chi(x\mathfrak{p})d\mathfrak{p} = \delta(x) \tag{6}$$

where $\Delta(\mathfrak{p}) = 1$ if $\mathfrak{p} = 0$ and $\Delta(\mathfrak{p}) = 0$ if $\mathfrak{p} \neq 0$. $\delta(x)$ is a delta function (discussed in the context of p-adic numbers in [6, 13]). It is not surprising that both $\Delta(\mathfrak{p})$ and $\delta(x)$ are outside the space of functions that we consider. A more general space (analogous to the rigged Hilbert space defined through a Gel'fand triplet) is required for them.

III. BASIC FORMALISM

Let $\mathcal{H}$ be the Hilbert space of complex functions $f(x)$ where $x \in \mathbb{Z}_p$ which are locally constant. The scalar product is given by

$$\langle f|g\rangle = \int_{\mathbb{Z}_p} [f(x)]^* \, g(x) \, dx \tag{7}$$

The Fourier transform is defined as

$$\tilde{f}(\mathfrak{p}) = \int_{\mathbb{Z}_p} dx\ \chi(-x\mathfrak{p}) f(x); \qquad \mathfrak{p} \in \mathbb{Q}_p/\mathbb{Z}_p \tag{8}$$

The inverse Fourier transform is given by

$$f(x) = \int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{p}\ \chi(x\mathfrak{p}) \tilde{f}(\mathfrak{p}); \qquad x \in \mathbb{Z}_p \tag{9}$$

We denote as $|X; x\rangle$ the position states and as $|P; \mathfrak{p}\rangle$ the momentum states. The X, P in the notation are not variables. They are used to indicate position and momentum states. The corresonding wavefunctions are delta functions and therefore they do not belong in the Hilbert space $\mathcal{H}$ but to the corresponding rigged Hilbert space.

IV. DISPLACEMENT OPERATORS AND THE HEISENBERG-WEYL GROUP

The phase space of this system is $\mathbb{Z}_p \times \mathbb{Q}_p/\mathbb{Z}_p$. Displacement operators in this phase space are defined as

$$D(\mathfrak{a}, b, \mathfrak{c}) = Z(\mathfrak{a}) X(b) \chi\left(\mathfrak{c} - \frac{1}{2}\mathfrak{a}b\right); \qquad \mathfrak{a}, \mathfrak{c} \in \mathbb{Q}_p/\mathbb{Z}_p; \qquad b \in \mathbb{Z}_p \tag{10}$$

where

$$\begin{aligned} Z(\mathfrak{a}) &= \int_{\mathbb{Z}_p} dx\ \chi(\mathfrak{a}x)\ |X; x\rangle\langle X; x| \\ X(b) &= \int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{p}\ \chi(-b\mathfrak{p})\ |P; \mathfrak{p}\rangle\langle P; \mathfrak{p}| \end{aligned} \tag{11}$$

The name displacement operators is related to the following properties:

$$Z(\mathfrak{a})|P; \mathfrak{p}\rangle = |P; \mathfrak{p} + \mathfrak{a}\rangle; \qquad X(b)|\mathsf{X}; x\rangle = |X; x + b\rangle \tag{12}$$

They are proved using Eq.(6). Using them we prove that

$$D(\mathfrak{a}, b, \mathfrak{c})\ D(\mathfrak{a}', b', \mathfrak{c}') = D[\mathfrak{a} + \mathfrak{a}', b + b', \mathfrak{c} + \mathfrak{c}' + 2^{-1}(\mathfrak{a}b' - \mathfrak{a}'b)] \tag{13}$$

Therefore the $D(\mathfrak{a}, b, \mathfrak{c})$ form a representation of the Heisenberg-Weyl group.

For later use we give the matrix elements of these operators:

$$\begin{aligned} \langle X; x_1|D(\mathfrak{a}, b, 0)|X; x_2\rangle &= \chi\left(\frac{1}{2}\mathfrak{a}b + \mathfrak{a}x_2\right) \delta(x_1 - x_2 - b) \\ \langle P; \mathfrak{p}_1|D(\mathfrak{a}, b, 0)|P; \mathfrak{p}_2\rangle &= \chi\left(-\frac{1}{2}\mathfrak{a}b - b\mathfrak{p}_2\right) \Delta(\mathfrak{p}_1 - \mathfrak{p}_2 - \mathfrak{a}) \end{aligned} \tag{14}$$

Eq.(14) is proved using Eq.(12).

V. DISPLACED PARITY OPERATORS

We first introduce the parity operator $P(0,0)$ with respect to the origin, through one of the following relations which are equivalent to each other:

$$P(0,0)|P;\mathfrak{p}\rangle = |P;-\mathfrak{p}\rangle; \qquad P(0,0)|X;x\rangle = |X;-x\rangle \tag{15}$$

As an example we consider the case $x = \overline{x}_0 + \overline{x}_1 p + \overline{x}_2 p^2 + ...$ where $0 \le \overline{x}_i \le p-1$. Then $-x = (p-\overline{x}_0) + (p-1-\overline{x}_1)p + (p-1-\overline{x}_2)p^2 + ...$ and

$$P(0,0)|X;\overline{x}_0 + \overline{x}_1 p + \overline{x}_2 p^2 + ...\rangle = |X;(p-\overline{x}_0) + (p-1-\overline{x}_1)p + (p-1-\overline{x}_2)p^2 + ...\rangle \tag{16}$$

The displaced parity operators (i.e., parity operators with respect to a point $(\mathfrak{a}, b)$ in the phase space $\mathbb{Z}_p \times \mathbb{Q}_p/\mathbb{Z}_p)$, are defined as

$$\begin{aligned} P(\mathfrak{a},b) &\equiv D(\mathfrak{a},b,0)P(0,0)[D(\mathfrak{a},b,0)]^\dagger \\ &= D(2\mathfrak{a},2b,0)P(0,0) = P(0,0)[D(2\mathfrak{a},2b,0)]^\dagger \end{aligned} \tag{17}$$

The displacement operators in the present context have properties analogous to the properties of the displacement operators for the harmonic oscillator (e.g.,[14] and references therein). The first is the marginal properties:

$$\begin{aligned} \int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{a}\; D(\mathfrak{a},b,0) &= |X;2^{-1}b\rangle\langle X;-2^{-1}b| \\ \int_{\mathbb{Z}_p} db\; D(\mathfrak{a},b,0) &= |P;2^{-1}\mathfrak{a}\rangle\langle P;-2^{-1}\mathfrak{a}| \\ \int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{a} \int_{\mathbb{Z}_p} db\; D(\mathfrak{a},b,0) &= P(0,0) \end{aligned} \tag{18}$$

They are proved by taking the matrix elements of both sides with respect to $\langle X;x_1|$ and $|X;x_2\rangle$, and using Eq.(14).

Using Eq.(18) we easily prove the following marginal relations for the displaced parity operators:

$$\begin{aligned} \int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{a}\; P(\mathfrak{a},b,0) &= |X;b\rangle\langle X;b| \\ \int_{\mathbb{Z}_p} db\; P(\mathfrak{a},b,0) &= |P;\mathfrak{a}\rangle\langle P;\mathfrak{a}| \\ \int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{a} \int_{\mathbb{Z}_p} db\; P(\mathfrak{a},b,0) &= \mathbf{1} \end{aligned} \tag{19}$$

The displaced parity operators are related to the displacement operators through a two-dimensional Fourier transform:

$$P(\mathfrak{a},b) = \int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{a}' \int_{\mathbb{Z}_p} db'\; D(\mathfrak{a}',b',0)\chi(\mathfrak{a}b' - \mathfrak{a}'b) \tag{20}$$

In order to prove this we multiply the third of Eqs(18) by $D(\mathfrak{a},b,0)$ on the left and by $[D(\mathfrak{a},b,0)]^\dagger$ on the right.

Another important property of the displacement operators is that for trace-class operators Θ acting on the Hilbert space $\mathcal{H}$

$$\int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{a} \int_{\mathbb{Z}_p} db\; D(\mathfrak{a},b,0)\;\Theta\;[D(\mathfrak{a},b,0)]^{\dagger} = [\mathrm{tr}\Theta]\;\mathbf{1} \tag{21}$$

We prove this by taking the matrix elements of both sides with respect to $\langle X;x_1|$ and $|X;x_2\rangle$, and then performing the integration.

Coherent states are defined as

$$|\mathfrak{a},b;s\rangle \equiv D(\mathfrak{a},b,0)\;|s\rangle;\qquad b\in\mathbb{Z}_p;\qquad \mathfrak{a}\in(\mathbb{Q}_p/\mathbb{Z}_p) \tag{22}$$

where $|s\rangle$ is an arbitrary state. Then Eq.(21) with $\Theta = |s\rangle\langle s|$, gives the resolution of the identity in terms of coherent states:

$$\int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{a} \int_{\mathbb{Z}_p} db\;|\mathfrak{a},b;s\rangle\langle\mathfrak{a},b;s| = \mathbf{1} \tag{23}$$

VI. WIGNER AND WEYL FUNCTIONS

Given a trace-class operator Θ the corresponding Wigner function $W(\Theta;\mathfrak{a},b)$ and the corresponding Weyl function $\widetilde{W}(\Theta;\mathfrak{a},b)$ are given by:

$$\begin{aligned} W(\Theta;\mathfrak{a},b) &= \mathrm{Tr}[\Theta\;P(\mathfrak{a},b,0)] = \chi(2\mathfrak{a}b)\int_{\mathbb{Z}_p} dx\chi(-2\mathfrak{a}x)\langle X;x|\Theta|X;2b-x\rangle \\ \widetilde{W}(\Theta;\mathfrak{a},b) &= \mathrm{Tr}[\Theta\;D(\mathfrak{a},b,0)] = \chi(2^{-1}\mathfrak{a}b)\int_{\mathbb{Z}_p} dx\chi(\mathfrak{a}x)\langle X;x|\Theta|X;x+b\rangle \end{aligned} \tag{24}$$

A direct consequence of Eq.(19) are the following marginal properties of the Wigner function:

$$\begin{aligned} \int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{a}\;W(\Theta;\mathfrak{a},b) &= \langle X;b|\Theta;|X;b\rangle \\ \int_{\mathbb{Z}_p} db\;W(\Theta;\mathfrak{a},b) &= \langle P;\mathfrak{a}|\Theta|P;\mathfrak{a}\rangle \\ \int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{a} \int_{\mathbb{Z}_p} db\;W(\Theta;\mathfrak{a},b) &= \mathrm{Tr}(\Theta) \end{aligned} \tag{25}$$

A direct consequence of Eq.(18) are the following marginal properties of the Weyl function:

$$\begin{aligned} \int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{a}\;\widetilde{W}(\Theta;\mathfrak{a},b) &= \langle X;-2^{-1}b|\Theta|X;2^{-1}b\rangle \\ \int_{\mathbb{Z}_p} db\;\widetilde{W}(\Theta;\mathfrak{a},b) &= \langle P;-2^{-1}\mathfrak{a}|\Theta|P;2^{-1}\mathfrak{a}\rangle \\ \int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{a} \int_{\mathbb{Z}_p} db\;\widetilde{W}(\Theta;\mathfrak{a},b) &= \mathrm{Tr}[\Theta P(0,0)] \end{aligned} \tag{26}$$

A direct consequence of Eq.(20) is that the Wigner function is related to the Weyl function through a two-dimensional Fourier transform:

$$W(\Theta;\mathfrak{a},b)=\int_{\mathbb{Q}_p/\mathbb{Z}_p}d\mathfrak{a}'\int_{\mathbb{Z}_p}db'\;\widetilde{W}(\Theta;\mathfrak{a}',b')\chi(\mathfrak{a}b'-\mathfrak{a}'b) \tag{27}$$

It is seen that the properties of the Wigner and Weyl functions are directly related to the properties of the displaced parity operators and the displacement operators, correspondingly.

We next show that an arbitrary trace-class operators Θ acting on the Hilbert space $\mathcal{H}$ can be expanded in terms of the displacement operators with the Weyl functions as coefficients; and also in terms of the displaced parity operators with the Wigner functions as coefficients:

$$\Theta=\int_{\mathbb{Q}_p/\mathbb{Z}_p}d\mathfrak{a}\int_{\mathbb{Z}_p}db\;D(\mathfrak{a},b,0)\widetilde{W}(\Theta;-\mathfrak{a},-b)=\int_{\mathbb{Q}_p/\mathbb{Z}_p}d\mathfrak{a}\int_{\mathbb{Z}_p}db\;P(\mathfrak{a},b,0)W(\Theta;\mathfrak{a},b) \tag{28}$$

We prove this by taking the matrix elements of both sides with respect to $\langle X;x_1|$ and $|X;x_2\rangle$. We then use Eqs(14), (24).

VII. DISCUSSION

Quantum systems on p-adic numbers have been studied in the literature in connection with physics at the Planck scale. We are interested in a very different application, namely the possibility of engineering such systems with potential applications to quantum information processing (cryptography, communications, etc). In ref.[11] we have studied a quantum system where the positions take values in $\mathbb{Z}_p$ and the momenta take values in $\mathbb{Q}_p/\mathbb{Z}_p$. We have discussed how we can engineer such a system from a semi-infinite chain of p-dimensional systems which are coupled in a particular way. This special coupling gives that system the p-adic structure.

In this paper we have continued this work and studied the properties of the displacement operators and the displaced parity operators and the corresponding properties of the Weyl and Wigner functions. The displacement operators are related to the displaced parity operators through the two-dimensional Fourier transform of Eq.(20); and correspondingly the Weyl and Wigner functions are related through the two-dimensional Fourier transform of Eq.(27). We have shown the marginal properties of the displacement operators in Eq.(18), and the corresponding marginal properties of the Weyl function in Eq.(26); also the marginal properties of the displacement parity operators in Eq.(19), and the corresponding marginal properties of the Wigner function in Eq.(25). We have also shown in Eq.(28) that an arbitrary trace-class operators Θ can be expanded in terms of the displacement operators with the Weyl functions as coefficients; and also in terms of the displaced parity operators with the Wigner functions as coefficients.

The work combines quantum mechanics with algebraic number theory.

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