Quantum mechanics on $\mathbb{Q}/\mathbb{Z}$

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Quantum mechanics with positions in $\mathbb{Q}/\mathbb{Z}$ and momenta in $\widehat{\mathbb{Z}}$ is considered. Displacement operators and coherent states, parity operators, Wigner and Weyl functions, and time evolution, are discussed. The restriction of the formalism to certain finite subspaces, is equivalent to Good's factorization of quantum mechanics on $\mathbb{Z}(q)$.

I. INTRODUCTION

Quantum mechanics and quantum field theory on the field $\mathbb{Q}_p$ of p-adic numbers have been studied by various authors (e.g., [1–12]). Applications to condensed matter physics have been discussed in [13–16]. Mathematical work related to these problems has been presented in [17–21]. General references on p-adic numbers are [22, 23], and on the related topic of profinite groups [24, 25].

Recently we have studied quantum mechanics with positions in $\mathbb{Z}_p$ (p-adic integers) and momenta in $\mathbb{Q}_p/\mathbb{Z}_p$, with emphasis on both the physical aspects [26], and the mathematical aspects (using inverse limits) [27]. Physical applications include the physics at the Planck scale, condensed matter, etc. Our interest is in quantum engineering of devices with p-adic arithmetic[26]. Such devices might have applications in the general area of information processing (e.g., [28]).

In the present paper we extend this work and study quantum mechanics with positions in $\mathbb{Q}/\mathbb{Z}$ and momenta in the Pontryagin dual group $\widehat{\mathbb{Z}}$ (where $\mathbb{Q}$ are the rational numbers, $\mathbb{Z}$ are the integers and $\widehat{\mathbb{Z}}$ is discussed below). We study the Heisenberg-Weyl group and discuss various properties of the displacement and parity operators, Wigner and Weyl functions and coherent states. Hamiltonians in this context, and the corresponding time evolution, are also studied. We show that a finite number of coupled finite quantum systems can be embedded into quantum mechanics on $\mathbb{Q}/\mathbb{Z}$ and in this sense there are many realistic physical systems, where the formalism is applicable.

There has been a lot of work recently (for reviews see [29–33]) on various aspects of finite quantum systems with positions and momenta in $\mathbb{Z}(d)$ (the integers modulo d). When $d = p^n$ (with p a fixed prime number), the large d limit of these systems is precisely the system with phase space $\mathbb{Z}_p \times (\mathbb{Q}_p/\mathbb{Z}_p)$ (studied in [26, 27]). When d takes all integer values, the large d limit of these systems is the system with phase space $\widehat{\mathbb{Z}} \times (\mathbb{Q}/\mathbb{Z})$ (studied here). This is shown in table I. The mathematical tools that bring these finite systems to the 'edge' are the inverse limit and direct limit. Using the inverse limit with the groups corresponding to momenta and the direct limit with the groups corresponding to positions, ensures the Pontryagin duality between the two groups for momenta and positions. The inverse limit of $\mathbb{Z}(p^n)$ is the profinite group $\mathbb{Z}_p$ and the direct limit of $\mathbb{Z}(p^n)$ is $\mathbb{Q}_p/\mathbb{Z}_p$ [27]. The inverse limit of $\mathbb{Z}(d)$ is the profinite group $\widehat{\mathbb{Z}}$, and the direct limit of $\mathbb{Z}(d)$ is $\mathbb{Q}/\mathbb{Z}$. In this paper we only mention the inverse limits very briefly (with reference to the literature) because the emphasis is on the physical aspects of these systems.

In section II, we introduce the $\mathbb{Q}/\mathbb{Z}$ and its Pontryagin dual group $\widehat{\mathbb{Z}}$. In section III we study quantum mechanics with positions in $\mathbb{Q}/\mathbb{Z}$ and momenta in $\widehat{\mathbb{Z}}$. We introduce the Schwartz-Bruhat space of functions on $\mathbb{Q}/\mathbb{Z}$ and their Fourier transforms, and we give various examples. In section IV we define the Heisenberg-Weyl group $HW(\mathbb{Q}/\mathbb{Z}, \widehat{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z})$. In section V we discuss Hamiltonians and time evolution in this context. In section VI we show that the quantum formalism for a finite number of coupled finite quantum systems can be embedded into quantum mechanics on $\mathbb{Q}/\mathbb{Z}$. In section VII we discuss displacement operators and coherent states, and in section VIII parity operators. In section IX we use them to define Wigner and Weyl functions and we study their properties. We conclude in section X with a discussion of our results.

II. PRELIMINARIES

The material in this section is known and we present it in order to explain the notation.

A. The groups $\mathbb{Z}_p$ and $\mathbb{Q}_p/\mathbb{Z}_p$

Elements in the field $\mathbb{Q}_p$ of p-adic numbers (where $p \in \Pi$, the set of prime numbers), can be written as

$$a_p = \sum_{\nu=\mathrm{ord}(a_p)}^{\infty} \overline{a}_\nu p^\nu; \qquad 0 \le \overline{a}_\nu \le p-1 \tag{1}$$

The absolute value of a_p is $|a_p|_p = p^{-\mathrm{ord}(a_p)}$. Elements in the ring $\mathbb{Z}_p$ of p-adic integers have $\mathrm{ord}(a_p) \ge 0$ and $|a_p|_p < 1$.

$\mathbb{Z}_p$ can be introduced as an inverse limit [24, 25]:

$$\varprojlim \mathbb{Z}(p^\ell) = \mathbb{Z}_p. \tag{2}$$

Therefore $\mathbb{Z}_p$ is a profinite group. The natural projection ξ_e from $\mathbb{Z}_p$ to $\mathbb{Z}(p^e)$ is given by

$$\xi_e(a_p) = \sum_{\nu=0}^{e-1} \overline{a}_\nu p^\nu \tag{3}$$

The Pontryagin dual group of $\mathbb{Z}_p$ is $\mathbb{Q}_p/\mathbb{Z}_p$ (the set of fractional p-adic numbers). Its elements are cosets and it is convinient to represent them with the element which has integer part equal to zero:

$$\mathfrak{b}_p = \overline{\mathfrak{b}}_{-k} p^{-k} + ... + \overline{\mathfrak{b}}_{-1} p^{-1}; \qquad 0 \le \overline{\mathfrak{b}}_i \le p-1 \tag{4}$$

The product $a_p \mathfrak{b}_p$ where $a_p \in \mathbb{Z}_p$ and $\mathfrak{b}_p \in \mathbb{Q}_p/\mathbb{Z}_p$ is also a coset in $\mathbb{Q}_p/\mathbb{Z}_p$. Additive characters in $\mathbb{Q}_p/\mathbb{Z}_p$ are given by

$$\chi_p(a_p \mathfrak{b}_p) = \exp(i2\pi a_p \mathfrak{b}_p) \tag{5}$$

B. The groups $\widehat{\mathbb{Z}}$ and $\mathbb{Q}/\mathbb{Z}$

$\widehat{\mathbb{Z}}$ can be introduced as the inverse limit [24, 25]

$$\varprojlim \mathbb{Z}(\ell) = \widehat{\mathbb{Z}} \tag{6}$$

and it can be shown that

$$\widehat{\mathbb{Z}} = \prod_{p \in \Pi} \mathbb{Z}_p. \tag{7}$$

$\widehat{\mathbb{Z}}$ is a profinite group. Its elements can be represented as sequences

$$s = (s_2, ..., s_p, ...); \quad s_p \in \mathbb{Z}_p; \quad p \in \Pi \tag{8}$$

where addition is performed componentwise. We next factorize an integer q in terms of prime numbers:

$$q = p_1^{e_1} ... p_\ell^{e_\ell}; \quad \Pi(q) = \{p_1, ..., p_\ell\}; \quad E(q) = \{e_1, ..., e_\ell\} \tag{9}$$

The natural projection π_q from $\widehat{\mathbb{Z}}$ to $\mathbb{Z}(q)$ is given by

$$\pi_q(s) = (\xi_{e_1}(s_{p_1}), ..., \xi_{e_\ell}(s_{p_\ell})); \quad p_j \in \Pi(q); \quad e_j \in E(q) \tag{10}$$

$\mathbb{Z}$ is embedded into $\widehat{\mathbb{Z}}$ by mapping $n \in \mathbb{Z}$ into $(n, n, n, ...) \in \widehat{\mathbb{Z}}$. For later use we introduce the following two subgroups of $\widehat{\mathbb{Z}}$:

$$\begin{aligned} \widehat{\mathbb{Z}}_{\text{odd}} &= \{(s_2, ..., s_p, ...) \mid s_p \in \mathbb{Z}_p; \quad |s_2|_2 = 1\} \\ \widehat{\mathbb{Z}}_{\text{even}} &= \{(s_2, ..., s_p, ...) \mid s_p \in \mathbb{Z}_p; \quad |s_2|_2 < 1\} \end{aligned} \tag{11}$$

The notation reflects the fact that the odd integers are embedded into $\widehat{\mathbb{Z}}_{\text{odd}}$ and the even integers into $\widehat{\mathbb{Z}}_{\text{even}}$. It is easily seen that $\widehat{\mathbb{Z}}_{\text{odd}} \cup \widehat{\mathbb{Z}}_{\text{even}} = \widehat{\mathbb{Z}}$.

The Pontryagin dual group of $\widehat{\mathbb{Z}}$ is $\mathbb{Q}/\mathbb{Z}$ and it is isomorphic to the direct sum $\sum_{p\in\Pi} \mathbb{Q}_p/\mathbb{Z}_p$. An element $\mathfrak{r} \in \mathbb{Q}/\mathbb{Z}$ can be written as $(\mathfrak{r}_2, ..., \mathfrak{r}_p, ...)$, where $\mathfrak{r}_p \in \mathbb{Q}_p/\mathbb{Z}_p$, and only a finite number of the $\mathfrak{r}_p$ are different from zero. Indeed let $\mathfrak{r} = r/q$ where r, q are coprime integers, and q is factorized as in Eq.(9). We can express r/q as

$$\frac{r}{q} = \frac{a_1}{p_1^{e_1}} + ... + \frac{a_\ell}{p_\ell^{e_\ell}} \tag{12}$$

and represent $\mathfrak{r}$ as $\mathfrak{r} = (\mathfrak{r}_2, ..., \mathfrak{r}_p, ...)$ where the $\mathfrak{r}_{p_i} = a_i p_i^{-e_i} \in \mathbb{Q}_{p_i}/\mathbb{Z}_{p_i}$ for the indices $p_1, ..., p_\ell$, and the rest $\mathfrak{r}_p$ are zero.

The product of $\mathfrak{r} = (\mathfrak{r}_2, ..., \mathfrak{r}_p, ...) \in \mathbb{Q}/\mathbb{Z}$ and $s = (s_2, ..., s_p, ...) \in \widehat{\mathbb{Z}}$ is $\mathfrak{r}s = (\mathfrak{r}_2 s_2, ..., \mathfrak{r}_p s_p, ...) \in \mathbb{Q}/\mathbb{Z}$. Additive characters in $\mathbb{Q}/\mathbb{Z}$ are given by

$$\chi(s\mathfrak{r}) = \prod_{p\in\Pi} \chi_p(s_p \mathfrak{r}_p) \tag{13}$$

This converges because only a finite number of the $\mathfrak{r}_p$ are different from zero.

C. Integrals

A complex function $f(a_p)$, where $a_p \in \mathbb{Q}_p$, has compact support with degree k if $f(a_p) = 0$ for $|a_p|_p > p^k$; and it is locally constant with degree n, if $f(a_p + b_p) = f(a_p)$ for $|b_p|_p \leq p^{-n}$. The Schwartz-Bruhat space consists of functions $f(a_p)$, where $a_p \in \mathbb{Q}_p$, which are locally constant and have compact support. In integrals over $\mathbb{Q}_p$ we use the Haar measure, normalized as:

$$\int_{\mathbb{Z}_p} ds_p = 1. \tag{14}$$

The Schwartz-Bruhat space S_p of complex functions over $\mathbb{Z}_p$, consists of locally constant functions $f(s_p)$, where $s_p \in \mathbb{Z}_p$ (these functions have compact support because $\mathbb{Z}_p$ is a compact group). Technical details about the integrals of locally constant functions over $\mathbb{Z}_p$, have been given in [26, 27]. The Fourier transform of a function in S_p which is locally constant with degree n , is given by

$$\widetilde{f}(\mathfrak{r}_p) = \int_{\mathbb{Z}_p} ds_p \chi_p(-s_p\mathfrak{r}_p) f(s_p); \qquad \mathfrak{r}_p \in \mathbb{Q}_p/\mathbb{Z}_p \tag{15}$$

and it has compact support with degree equal to n (these functions are locally constant because they are defined in $\mathbb{Q}_p/\mathbb{Z}_p$). Technical details about the integrals of functions with compact support over $\mathbb{Q}_p/\mathbb{Z}_p$, have been given in [26, 27]. The inverse Fourier transform is given by

$$f(s_p) = \int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{r}_p \chi_p(s_p\mathfrak{r}_p) \widetilde{f}(\mathfrak{r}_p); \qquad s_p \in \mathbb{Z}_p. \tag{16}$$

For later use we define the function

$$\begin{aligned} \Delta_p(\mathfrak{r}_p) &= 0; \text{ if } \mathfrak{r}_p \neq 0; \qquad \mathfrak{r}_p \in \mathbb{Q}_p/\mathbb{Z}_p \\ \Delta_p(0) &= 1. \end{aligned} \tag{17}$$

where the zero in $\mathbb{Q}_p/\mathbb{Z}_p$ is the coset with all the p-adic integers. We also define the function $\delta_p(s_p)$ which is equal to zero when $x \neq 0$ and for which

$$\int_{\mathbb{Z}_p} ds_p \delta_p(s_p) f(s_p) = f(0); \qquad \mathfrak{r}_p \in \mathbb{Q}_p/\mathbb{Z}_p. \tag{18}$$

This is a generalized function (it is not locally constant and therefore it does not belong to the Schwartz-Bruhat space). Generalized functions in the present context, are discussed rigorously in [21]. Then

$$\begin{aligned} \int_{\mathbb{Z}_p} ds_p \chi_p(-s_p\mathfrak{r}_p) &= \Delta_p(\mathfrak{r}_p); \qquad \mathfrak{r}_p \in \mathbb{Q}_p/\mathbb{Z}_p \\ \int_{\mathbb{Q}_p/\mathbb{Z}_p} d\mathfrak{r}_p \chi_p(-s_p\mathfrak{r}_p) &= \delta_p(s_p); \qquad s_p \in \mathbb{Z}_p. \end{aligned} \tag{19}$$

III. THE SCHWARTZ-BRUHAT SPACE S

In this section we study quantum mechanics with positions in $\mathbb{Q}/\mathbb{Z}$ and momenta in $\widehat{\mathbb{Z}}$. The wavefunctions in the momentum representation $F(s)$ and in the position representation $f(\mathfrak{r})$ (where $s \in \widehat{\mathbb{Z}}$ and $\mathfrak{r} \in \mathbb{Q}/\mathbb{Z}$) belong to the Schwartz-Bruhat space **S** which is defined below.

Definition III.1. The Schwartz-Bruhat space **S**[17–20], is the space of finite linear combinations of complex functions $F(s)$ such that

$$F(s) = \prod_{p \in \Pi} F_p(s_p); \qquad s = (s_2, ..., s_p, ...) \in \widehat{\mathbb{Z}}; \quad s_p \in \mathbb{Z}_p \tag{20}$$

where

(1) $F_p(s_p)$ are locally constant complex functions,

(2) $F_p(s_p) = 1$ for all but a finite number of $p \in \Pi$. We denote as $\Pi[F(s)]$ the finite subset of Π, with the indices for which $F_p(s_p) \neq 1$.

Integrals over $\widehat{\mathbb{Z}}$ of these functions are given by the finite product

$$\int_{\widehat{\mathbb{Z}}} F(s)ds = \prod_{p\in\Pi[F(s)]} \int_{\mathbb{Z}_p} F_p(s_p)ds_p. \tag{21}$$

The Fourier transform is given by

$$f(\mathfrak{r}) = \int_{\widehat{\mathbb{Z}}} ds\ \chi(-s\mathfrak{r})F(s); \qquad \mathfrak{r}\in\mathbb{Q}/\mathbb{Z} \tag{22}$$

It is known that the Schwartz-Bruhat space $\mathbf{S}$ is invariant under Fourier transforms[17, 19], i.e., $f(\mathfrak{r})\in\mathbf{S}$. An alternative definition of the space $\mathbf{S}$ in terms of functions $f(\mathfrak{r})$ where $\mathfrak{r}\in\mathbb{Q}/\mathbb{Z}$ is as follows:

Definition III.2. The Schwartz-Bruhat space $\mathbf{S}$, is the space of finite linear combinations of complex functions $f(\mathfrak{r})$ such that

$$f(\mathfrak{r}) = \prod_{p\in\Pi} f_p(\mathfrak{r}_p); \qquad \mathfrak{r}=(\mathfrak{r}_2,...,\mathfrak{r}_p,...)\in\mathbb{Q}/\mathbb{Z}; \quad \mathfrak{r}_p\in\mathbb{Q}_p/\mathbb{Z}_p, \tag{23}$$

where

(1) $f_p(\mathfrak{r}_p)$ are complex functions with compact support,

(2) $f_p(\mathfrak{r}_p)=\Delta_p(\mathfrak{r}_p)$ for all but a finite number of $p\in\Pi$. We denote as $\Pi[f(\mathfrak{r})]$ the finite subset of Π, with the indices for which $f_p(\mathfrak{r}_p)\neq\Delta_p(\mathfrak{r}_p)$.

The requirement that the $F_p(s_p)$ are locally constant complex functions (in the first definition), corresponds through Fourier transform to the requirement that the $f_p(\mathfrak{r}_p)$ are complex functions with compact support (in the second definition). It can be proved [27] that if a function is locally constant with degree n and has constant support with degree k, then its Fourier transform is locally constant with degree k and has constant support with degree n. Also the requirement that $F_p(s_p)=1$ for all but a finite number of $p\in\Pi$, corresponds through Fourier transform to the requirement that $f_p(\mathfrak{r}_p)=\Delta_p(\mathfrak{r}_p)$ for all but a finite number of $p\in\Pi$.

The Schwartz-Bruhat space $\mathbf{S}$, is isomorphic to the restricted tensor product of the Schwartz-Bruhat spaces S_p:

$$\mathbf{S} = \bigotimes_{p\in\Pi}{}' S_p \tag{24}$$

The prime in the notation, indicates that it is a restricted tensor product. The condition 'restricted' is related to condition (2) in the above definitions, and it is needed for the convergence of the products (20), (23).

Physically linear combinations of factorizable functions like in Eqs(20),(23), represent entangled states with respect to this particular factorization, where a system with positions in $\mathbb{Q}/\mathbb{Z}$ and momenta in $\widehat{\mathbb{Z}}$ is factorized in terms of component systems with positions in $\mathbb{Q}_p/\mathbb{Z}_p$ and momenta in $\mathbb{Z}_p$.

The integral over $\mathbb{Q}/\mathbb{Z}$, of a complex function $f(\mathfrak{r})\in\mathbf{S}$ is given by

$$\int_{\mathbb{Q}/\mathbb{Z}} f(\mathfrak{r})d\mathfrak{r} = \prod_{p\in\Pi[f(\mathfrak{r})]} \int_{\mathbb{Q}_p/\mathbb{Z}_p} f_p(\mathfrak{r}_p)d\mathfrak{r}_p \tag{25}$$

The inverse Fourier transform is given by

$$F(s)=\int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{x}\chi(s\mathfrak{x})f(\mathfrak{x});\qquad s\in\widehat{\mathbb{Z}} \tag{26}$$

For $\lambda\in\mathbb{Z}$, a change in the variables $\mathfrak{x}=\lambda\mathfrak{x}'$ in the integral of Eq.(25), or $s=\lambda s'$ in the integral of Eq.(21), where $\lambda\in\mathbb{Q}$, is performed as follows:

$$d\mathfrak{x}=\prod_{p\in\Pi} d\mathfrak{x}_p=\prod_{p\in\Pi}|\lambda|_p\; d\mathfrak{x}'_p=\frac{1}{|\lambda|_\infty}d\mathfrak{x}';\qquad ds=\prod_{p\in\Pi} ds_p=\prod_{p\in\Pi}|\lambda|_p\; ds'_p=\frac{1}{|\lambda|_\infty}ds' \tag{27}$$

where $|\lambda|_\infty$ denotes the 'usual' absolute value. In order to prove this we first prove that in integrals over $\mathbb{Q}_p/\mathbb{Z}_p$ a change in the variables $\mathfrak{x}_p=\lambda\mathfrak{x}'_p$ is performed with $d\mathfrak{x}_p=|\lambda|_p\; d\mathfrak{x}'_p$. We then use the Ostrowski relation $|\lambda|_\infty\prod_{p\in\Pi}|\lambda|_p=1$.

The scalar product of the functions $F(s)$, $G(s)$, is given by

$$(F,G)=\int_{\widehat{\mathbb{Z}}}\overline{F(s)}G(s)ds. \tag{28}$$

The scalar product of the functions $f(\mathfrak{x})$, $g(\mathfrak{x})$ is given by

$$(f,g)=\int_{\mathbb{Q}/\mathbb{Z}}\overline{f(\mathfrak{x})}g(\mathfrak{x})d\mathfrak{x}. \tag{29}$$

The following relation is useful in calculations:

$$\int_{\widehat{\mathbb{Z}}} ds\;\chi(s\mathfrak{x})=\Delta(\mathfrak{x}) \tag{30}$$

where $\Delta(\mathfrak{x})=\prod\Delta_p(\mathfrak{x}_p)$, i.e.,

$$\begin{aligned}\Delta(\mathfrak{x})&=0;\;\text{ if }\;\mathfrak{x}\neq 0;\qquad \mathfrak{x}\in\mathbb{Q}/\mathbb{Z}\\ \Delta(0)&=1\end{aligned} \tag{31}$$

We note here that the zero in $\mathbb{Q}/\mathbb{Z}$ is the coset with all the integers. Taking into account Eq.(27), we prove that

$$\int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{x}\;\Delta(\lambda\mathfrak{x}-\lambda\mathfrak{a})f(\mathfrak{x})=|\lambda|_\infty f(\mathfrak{a});\qquad \lambda\in\mathbb{Z}. \tag{32}$$

Another useful relation is:

$$\int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{x}\;\chi(s\mathfrak{x})=\delta(s);\qquad s\in\widehat{\mathbb{Z}} \tag{33}$$

where $\delta(s)=\prod\delta_p(s_p)$.

A. Examples

In the examples below we restrict the formalism to functions of various types. We calculate their Fourier transforms and show that we get quantum mechanics with variables in various subgroups of $\mathbb{Q}/\mathbb{Z}$

and in their Pontryagin dual groups. We use the notation $\mathbb{Q}^{(p_1,...,p_N)}/\mathbb{Z}$ for the additive subgroup of $\mathbb{Q}/\mathbb{Z}$ comprised of rational numbers (modulo integers) of the type r/q where r, q are coprime integers and q is factorized in terms of prime numbers as $q = p_1^{e_1}....p_N^{e_N}$.

We consider functions of the type

$$F(s) = F_{p_1}(s_{p_1})...F_{p_\ell}(s_{p_\ell}). \tag{34}$$

Their Fourier transform $f(\mathfrak{x})$ is

$$f(\mathfrak{x}) = \prod_{j=1}^{\ell}\left[\int_{\mathbb{Q}_{p_j}/\mathbb{Z}_{p_j}} d\mathfrak{x}_{p_j}\chi_{p_j}(-s_{p_j}\mathfrak{x}_{p_j})F_{p_j}(s_{p_j})\right]\prod_{p\in\Pi_1}\Delta_p(\mathfrak{x}_p); \qquad \Pi_1 = \Pi - \{p_1, ..., p_\ell\}. \tag{35}$$

It is seen that $f(\mathfrak{x})$ can take non-zero values, only if

$$\mathfrak{x} \in \mathbb{Q}^{(p_1,...,p_\ell)}/\mathbb{Z}. \tag{36}$$

Indeed, in this case any p-adic representation of $\mathfrak{x}$ with $p \in \Pi_1$, is a p-adic integer, i.e., it belongs to the 'zero coset' of $\mathbb{Q}_p/\mathbb{Z}_p$ and $\Delta_p(\mathfrak{x}_p) = 1$.

We restrict the formalism further, with the requirement that the functions $F_{p_1}(s_{p_1})$,...,$F_{p_\ell}(s_{p_\ell})$, are locally constant with given degrees $e_1, ..., e_\ell$. This is clearly a further restriction because earlier these functions were locally constant with any degree, but here we require that they are locally constant with given degrees. In this case it is sufficient for s to take values on the projection of $\widehat{\mathbb{Z}}$ into $\mathbb{Z}(q)$ with $q = p_1^{e_1}...p_\ell^{e_\ell}$ (defined in Eq.(10)). Indeed, the variables s_{p_j} are elements of $\mathbb{Z}_{p_j}/p_j^{e_j}\mathbb{Z}_{p_j} \cong \mathbb{Z}(p_j^{e_j})$ and then

$$s = (s_{p_1}, ..., s_{p_\ell}) \in \mathbb{Z}(p_1^{e_1}) \times ... \times \mathbb{Z}(p_\ell^{e_\ell}) \cong \mathbb{Z}(q) \tag{37}$$

Then the Fourier transform $f_{p_j}(\mathfrak{x}_{p_j})$, of $F_{p_j}(s_{p_j})$, has compact support with degree e_j. Therefore we can regard the variables $\mathfrak{x}_{p_j}$ as elements of $p_j^{-e_j}\mathbb{Z}_{p_j}/\mathbb{Z}_{p_j} \cong \mathbb{Z}(p_j^{e_j})$, and the variable $\mathfrak{x}$ as an element of $\mathbb{Z}(q)$.

It is seen that if we restrict the present formalism into a subspace of $\mathbf{S}$ comprised of functions like in Eq.(34) where $F_{p_1}(s_{p_1})$,...,$F_{p_\ell}(s_{p_\ell})$, are locally constant with given degrees $e_1, ..., e_\ell$, then we get a formalism which is mathematically equivalent to quantum mechanics in $\mathbb{Z}(q)$. We discuss this point in more detail later, in connection with Good's factorization.

IV. THE HEISENBERG-WEYL GROUP HW($\mathbb{Q}/\mathbb{Z}, \widehat{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z}$)

Definition IV.1. The displacement operators $D(\mathfrak{a}, b, \mathfrak{c})$ act on the functions $F(s) \in \mathbf{S}$, where $\mathfrak{a}, \mathfrak{c} \in \mathbb{Q}/\mathbb{Z}$ and $b, s \in \widehat{\mathbb{Z}}$, as follows:

$$[D(\mathfrak{a}, b, \mathfrak{c})F](s) = \chi(\mathfrak{c} - \mathfrak{a}b + 2\mathfrak{a}s)\, F(s - b). \tag{38}$$

We use 'fraktur' characters for elements of $\mathbb{Q}/\mathbb{Z}$ and ordinary characters for elements of $\widehat{\mathbb{Z}}$. It is easily seen that $D(\mathfrak{a} + 1, b, \mathfrak{c}) = D(\mathfrak{a}, b, \mathfrak{c})$ and this is consistent with the fact that $\mathfrak{a} \in \mathbb{Q}/\mathbb{Z}$.

Proposition IV.2.

(1) The $D(\mathfrak{a}, b, \mathfrak{c})$ form a representation of the Heisenberg-Weyl group (for which we use the notation $\mathbf{HW}(\mathbb{Q}/\mathbb{Z}, \widehat{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z})$ *to indicate that $\mathfrak{a}, \mathfrak{c} \in \mathbb{Q}/\mathbb{Z}$ and $b \in \widehat{\mathbb{Z}}$).*

(2) The displacement operators $D(\mathfrak{a}, b, \mathfrak{c})$ act on the functions $f(\mathfrak{x}) \in \mathbf{S}$, where $\mathfrak{x} \in \mathbb{Q}/\mathbb{Z}$, as follows:

$$[D(\mathfrak{a}, b, \mathfrak{c})f](\mathfrak{x}) = \chi(\mathfrak{c} + \mathfrak{a}b - \mathfrak{x}b)\, f(\mathfrak{x} - 2\mathfrak{a}). \tag{39}$$

(3)

$$[D(\mathfrak{a}, b, \mathfrak{c})]^\dagger = D(-\mathfrak{a}, -b, -\mathfrak{c}); \qquad D(\mathfrak{a}, b, \mathfrak{c})[D(\mathfrak{a}, b, \mathfrak{c})]^\dagger = \mathbf{1}. \tag{40}$$

Proof.

(1) The Heisenberg-Weyl group has elements $g(a, b, c)$ and the multiplication rule:

$$g(a_1, b_1, c_1)g(a_2, b_2, c_2) = g(a_1 + a_2, b_1 + b_2, c); \qquad c = c_1 + c_2 + (a_1 b_2 - a_2 b_1)\,. \tag{41}$$

We use the definition of Eq.(38) and prove that the $D(\mathfrak{a}, b, \mathfrak{c})$ obey this multiplication rule. The inverse of $D(\mathfrak{a}, b, \mathfrak{c})$ is $D(-\mathfrak{a}, -b, -\mathfrak{c})$. This proves the statement.

(2) We insert the Fourier transform of Eq.(22) into the left hand side of Eq.(39) and use Eq.(38). This leads to the right hand side of Eq.(39).

(3) It is easily seen that for any $g(\mathfrak{x}), f(\mathfrak{x}) \in \mathbf{S}$

$$(D(\mathfrak{a}, b, \mathfrak{c})g, f) = (g, D(-\mathfrak{a}, -b, -\mathfrak{c})f) \tag{42}$$

and also that

$$(D(\mathfrak{a}, b, \mathfrak{c})g, D(\mathfrak{a}, b, \mathfrak{c})f) = (g, f) \tag{43}$$

□

V. HAMILTONIANS AND TIME EVOLUTION

In the harmonic oscillator context we consider the operators

$$\Theta_n(\alpha, \beta) = \exp(-i\beta \hat{x}^n) \exp(i\alpha \hat{p}^n) \tag{44}$$

where $\hat{x}$ and $\hat{p}$ are the usual position and momentum operators. This operator acts on a harmonic oscillator wavefunction $g(x)$ as follows:

$$[\Theta_n(\alpha, \beta)g](x) = \frac{1}{2\pi} \exp(-i\beta x^n) \int_{\mathbb{R}} dp \left[\exp(i\alpha p^n - ipx) \int_{\mathbb{R}} dx' \exp(ipx') g(x')\right] \tag{45}$$

We assume here that the integrals converge. The exponentials of a very wide class of harmonic oscillator Hamiltonians h can be written as

$$\exp(ih) = \prod_n \Theta_n(\alpha, \beta) \tag{46}$$

If we calculate the eigenvalues and eigenvectors of these operators, then we can calculate the time evolution of the system.

This formalism is generalized into our context. We first introduce operators analogous to $\Theta_n(\alpha, \beta)$.

Definition V.1. The operator $\theta_n(\mathfrak{a}, b)$ where $\mathfrak{a} \in \mathbb{Q}/\mathbb{Z}$, $b \in \widehat{\mathbb{Z}}$ and $n \in \mathbb{Z}^+$, acts on the functions $f(\mathfrak{x}) \in \mathbf{S}$ where $\mathfrak{x} \in \mathbb{Q}/\mathbb{Z}$, as follows:

$$[\theta_n(\mathfrak{a}, b)f](\mathfrak{x}) = \chi(-b\mathfrak{x}^n) \int_{\widehat{\mathbb{Z}}} ds \left[\chi\left(2\mathfrak{a}s^n - s\mathfrak{x}\right) \int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{x}' \chi\left(s\mathfrak{x}'\right) f(\mathfrak{x}')\right]. \tag{47}$$

Through Fourier transform, it is easily seen that the operator $\theta_n(\mathfrak{a}, b)$ acts on functions $F(s) \in \mathbf{S}$, where $s \in \widehat{\mathbb{Z}}$, as follows:

$$[\theta_n(\mathfrak{a}, b)F](s) = \int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{x} \left[\chi\left(-b\mathfrak{x}^n + s\mathfrak{x}\right) \int_{\widehat{\mathbb{Z}}} ds' \chi(2\mathfrak{a}s'^n - s'\mathfrak{x})F(s') \right]. \tag{48}$$

In the special case $n = 1$, we get

$$\theta_1(\mathfrak{a}, b) = \chi\left(-\mathfrak{a}b\right) D(\mathfrak{a}, b, 0) \tag{49}$$

When $\mathfrak{a} = 0$, Eq.(47) reduces to

$$[\theta_n(0, b)f](\mathfrak{x}) = \chi\left(-b\mathfrak{x}^n\right) f(\mathfrak{x}), \tag{50}$$

and when $b = 0$, Eq.(48) reduces to

$$[\theta_n(\mathfrak{a}, 0)F](s) = \chi\left(2\mathfrak{a}s^n\right) F(s). \tag{51}$$

The exponentials of a very wide class of Hamiltonians h can be written as the operator

$$\exp(ih) = \prod_n \theta_n(\mathfrak{a}_n, b_n) \tag{52}$$

We can define real powers of this operator, through its eigenvalues and eigenvectors. Then the time evolution operator is $U(t) = [\exp(ih)]^t$ where t is a real number.

For practical numerical calculations we can use truncations and work in subspaces of $\mathbf{S}$. We consider functions $F(s) = \prod F_p(s_p)$ and we assume that all functions $F_p(s_p)$ are locally constant with degree less or equal to n_p. Also let $\Pi_{\rm cut}$ be the finite set of primes which are smaller than some large 'cutoff prime'. Clearly a good approximation will require large 'cutoff prime' and large values of n_p. Then $F(s)$ can be written as a complex vector with dimension

$$\ell = \prod_{p \in \Pi_{\rm cut}} p^{n_p} \tag{53}$$

The Fourier transforms of these functions are $f(\mathfrak{x}) = \prod f_p(\mathfrak{x}_p)$ where $f_p(\mathfrak{x}_p)$ have compact support with degree less or equal to n_p, and $f(\mathfrak{x})$ can also be written as a ℓ-dimensional complex vector. Then Eqs(47), (48) reduce to matrix equations with $\ell \times \ell$ complex matrices. The $\exp(ih)$ also can be written as a $\ell \times \ell$ complex matrix M and then the time evolution operator is $U(t) = M^t$ where t is the time which is a real variable. There is a multivaluedness associated with real powers of matrices (because they are defined through logarithms), but principal values can be considered. We do not pursue this direction further, in this paper.

VI. EMBEDDING OF THE FORMALISM FOR FINITE QUANTUM SYSTEMS INTO QUANTUM MECHANICS ON $\mathbb{Q}/\mathbb{Z}$

Good factorized Fourier transforms [34–36] in the context of 'fast Fourier transforms' and this has been extended to a factorization of quantum mechanics on $\mathbb{Z}(q)$ in [37, 38]. We discuss this briefly below. We then restrict quantum mechanics on $\mathbb{Q}/\mathbb{Z}$ which is described by the space $\mathbf{S}$, into a q-dimensional subspace $\mathbf{S}(q)$ (defined below) and show that we get Good's factorization of quantum mechanics on $\mathbb{Z}(q)$. In this sense quantum mechanics on $\mathbb{Q}/\mathbb{Z}$ is a generalization of Good's factorization (and they both use the Chinese remainder theorem). We also show that the formalism for a finite number of coupled finite quantum systems can be embedded into quantum mechanics on $\mathbb{Q}/\mathbb{Z}$.

A. Factorization of quantum mechanics on $\mathbb{Z}(q)$

Good has factorized the Fourier transform on $\mathbb{Z}(q)$ where $q = q_1 ... q_\ell$ and any two of the q_i are coprime, in terms of 'smaller' Fourier transforms on $\mathbb{Z}(q_i)$. This factorization is based on the Chinese remainder theorem and we present it for the case that q is factorized in terms of powers of prime numbers. We use the notation of Eq.(9) and we define the

$$u_i = \frac{q}{p_i^{e_i}}; \quad t_i u_i = 1 \ (\mathrm{mod}\ p_i^{e_i}); \quad t_i \in \mathbb{Z}(p_i^{e_i}). \tag{54}$$

We also define the $w_i = t_i u_i \in \mathbb{Z}(q)$. The fact that t_i is the inverse of u_i in $\mathbb{Z}(p_i^{e_i})$, implies that w_i is an integral multiple of $p_i^{e_i}$ plus 1. We will use the notation

$$\omega(n; \alpha) = \exp\left(\frac{i2\pi\alpha}{n}\right); \quad \alpha \in \mathbb{Z}(n) \tag{55}$$

for the roots of unity. A useful relation is that

$$i \neq j \quad \rightarrow \quad \omega(q; w_i u_j) = 1 \tag{56}$$

We then consider the isomorphism

$$\mathbb{Z}(q) \cong \mathbb{Z}(p_1^{e_1}) \times ... \times \mathbb{Z}(p_\ell^{e_\ell}) \tag{57}$$

Good introduced two bijective maps between these groups. The first one (which he called 'Sino correspondence') is:

$$\begin{aligned} & s \ \leftrightarrow \ (s_1, ..., s_\ell); \quad s \in \mathbb{Z}(q); \quad s_i \in \mathbb{Z}(p_i^{e_i}) \\ & s_i = s \ (\mathrm{mod}\ p_i^{e_i}); \quad s = \sum_i s_i w_i; \quad w_i = t_i u_i \in \mathbb{Z}(q) \end{aligned} \tag{58}$$

The second one (which he called 'Ruritanian correspondence') is:

$$\begin{aligned} & r \ \leftrightarrow \ (\hat{r}_1, ..., \hat{r}_\ell); \quad r \in \mathbb{Z}(q); \quad \hat{r}_i \in \mathbb{Z}(p_i^{e_i}) \\ & \hat{r}_i = r t_i \ (\mathrm{mod}\ p_i^{e_i}); \quad r = \sum_i \hat{r}_i u_i. \end{aligned} \tag{59}$$

Good proved (using Eq.(56)) that

$$\omega(q; rs) = \omega\left(p_1^{e_1}; \hat{r}_1 s_1\right) ... \omega\left(p_\ell^{e_\ell}; \hat{r}_\ell s_\ell\right) \tag{60}$$

In [37, 38] we have extended Good's scheme and showed that various quantities in quantum mechanics in $\mathbb{Z}(q)$ are factorized in terms of their counterparts in quantum mechanics on $\mathbb{Z}(p_i^{e_i})$ (with $i = 1, ..., \ell$). We use the notation $|P; s\rangle$ and $|X; r\rangle$ for the momentum and position bases in the q-dimensional Hilbert space H of quantum mechanics on $\mathbb{Z}(q)$. Here $s, r \in \mathbb{Z}(q)$ (the P and X in the notation are not variables, but they simply indicate momentum and position states). We also use the notation $|P_i; s_i\rangle$ and $|X_i; r_i\rangle$ for the momentum and position bases in the $p_i^{e_i}$-dimensional space H_i of quantum mechanics on $\mathbb{Z}(p_i^{e_i})$ (here $s_i, r_i \in \mathbb{Z}(p_i^{e_i})$). H is isomorphic to the tensor product of H_i, and we introduce a bijective map between them, with the following map between the momentum bases in the two spaces:

$$|P; s\rangle \ \leftrightarrow \ |P_1; s_1\rangle \otimes ... \otimes |P_\ell; s_\ell\rangle; \quad s \in \mathbb{Z}(q); \quad s_i \in \mathbb{Z}(p_i^{e_i}) \tag{61}$$

We then use Eq.(60) to show that the corresponding map between the position bases in the two spaces is:

$$|X;r\rangle \;\leftrightarrow\; |X_1;\hat{r}_1\rangle \otimes ... \otimes |X_\ell;\hat{r}_\ell\rangle; \quad r\in\mathbb{Z}(q); \quad \hat{r}_i \in \mathbb{Z}(p_i^{e_i}) \tag{62}$$

We note that the map of Eq.(58) is used for the momentum states, and the map of Eq.(59) is used for the position states. Therefore the wavefunctions in the momentum representation in H are finite linear combinations of functions of the form

$$F(s)=F_1(s_1)...F_\ell(s_\ell); \quad s\in\mathbb{Z}(q); \quad s_i\in\mathbb{Z}(p_i^{e_i}), \tag{63}$$

where $F_i(s_i)$ are in H_i. Similarly, the wavefunctions in the position representation in H, are finite linear combinations of functions of the form

$$f(r)=f_1(\hat{r}_1)...f_\ell(\hat{r}_\ell); \quad r\in\mathbb{Z}(q); \quad \hat{r}_i\in\mathbb{Z}(p_i^{e_i}). \tag{64}$$

where $f_i(\hat{r}_i)$ are in H_i. In a physical terminology we can say that a quantum system with q-dimensional Hilbert space is regarded as an ℓ-partite system, where the i-component system has $p_i^{e_i}$-dimensional Hilbert space.

We have also shown that the displacement operators factorize in terms of displacement operators in the component systems.

B. Restriction of the formalism on S to its subspace S(q)

Definition VI.1. $\mathbf{S}(q)$ is the q-dimensional subspace of $\mathbf{S}$ which consists of finite linear combinations of complex functions of the form of Eq.(20) such that

(1) $F_p(s_p)=1$ if $p\notin\Pi(q)$ (in the notation of Eq.(9)),

(2) $F_{p_i}(s_{p_i})$ where $p_i\in\Pi(q)$, is a locally constant function with degree $e_i\in E(q)$

Equivalently, $\mathbf{S}(q)$ is the subspace of $\mathbf{S}$ which consists of finite linear combinations of complex functions of the form of Eq.(23) such that

(1) $f_p(\mathfrak{x}_p)=\Delta_p(\mathfrak{x}_p)$ if $p\notin\Pi(q)$,

(2) $f_{p_i}(\mathfrak{x}_{p_i})$ where $p_i\in\Pi(q)$, has compact support with degree $e_i\in E(q)$

The dimension of $\mathbf{S}(q)$ is $\prod p_i^{e_i}$ where $p_i\in\Pi(q)$ and $e_i\in E(q)$.

It is now clear that functions in $\mathbf{S}(q)$ can be regarded as finite linear combinations of factorizable functions $F(s)$, where

$$s=(s_{p_1},...,s_{p_\ell})\in[\mathbb{Z}_{p_1}/p_1^{e_1}\mathbb{Z}_{p_1}]\times...\times[\mathbb{Z}_{p_\ell}/p_\ell^{e_\ell}\mathbb{Z}_{p_\ell}]\cong\mathbb{Z}(p_1^{e_1})\times...\times\mathbb{Z}(p_\ell^{e_\ell}). \tag{65}$$

Therefore in the subspace $\mathbf{S}(q)$, the function $F(s)$ is analogous to the product of Eq.(63), in Good's formalism. Similarly functions in $\mathbf{S}(q)$ can be regarded as finite linear combinations of factorizable functions $f(\mathfrak{x})$, where

$$\mathfrak{x}=(\mathfrak{x}_{p_1},...,\mathfrak{x}_{p_\ell})\in\left[p_1^{-e_1}\mathbb{Z}_{p_1}/\mathbb{Z}_{p_1}\right]\times...\times\left[p_\ell^{-e_\ell}\mathbb{Z}_{p_\ell}/\mathbb{Z}(p_\ell^{e_\ell})\right]\cong\mathbb{Z}(p_1^{e_1})\times...\times\mathbb{Z}(p_\ell^{e_\ell}). \tag{66}$$

Therefore in the subspace $\mathbf{S}(q)$, the function $f(\mathfrak{x})$ is analogous to the product of Eq.(64), in Good's formalism.

Furthermore, we can easily see that in the subspace $\mathbf{S}(q)$, the product of characters in the right hand side of Eq.(13), reduces to the product of characters in the right hand side of the Good factorization in Eq.(60).

We next consider N coupled finite quantum systems described by spaces with dimensions $q_1, ..., q_N$ (where the (q_i, q_j) are not necessarily coprime). The positions in this system belong in $\mathbb{Z}(q_1) \times ... \times \mathbb{Z}(q_N)$. Using any bijective map between $\mathbb{Z}(q)$ and $\mathbb{Z}(q_1) \times ... \times \mathbb{Z}(q_N)$ (where $q = q_1...q_N$) we describe this system as a finite system with positions r and momenta s in $\mathbb{Z}(q)$ and with the space $\mathbf{S}(q)$. The next step is to use Good's formalism to factorize the whole system into 'mathematical subsystems' which are different from the 'physical subsystems' (in general the number of physical subsystems N is different from the number of mathematical subsystems ℓ). We factorize q as in Eq.(9) and replace the momentum s with $(s_1, ..., s_\ell)$ and the position r with $(\hat{r}_1, ..., \hat{r}_\ell)$, using the bijective maps of Eqs.(58),(59), correspondingly. Then the wavefunctions in the momentum representation are finite linear combinations of $\prod F_{p_i}(s_i)$, and in the position representation they are finite linear combinations of $\prod f_{p_i}(\hat{r}_i)$. This shows that the quantum formalism for a finite number of coupled finite quantum systems can be embedded into quantum mechanics on $\mathbb{Q}/\mathbb{Z}$.

VII. PROPERTIES OF THE DISPLACEMENT OPERATORS AND COHERENT STATES

We consider a trace class operator θ with kernel $\theta(s, s')$. This operator acts on functions $F(s) \in \mathbf{S}$ as

$$[\theta F](s) = \int_{\widehat{\mathbb{Z}}} ds' \theta(s, s') F(s'). \tag{67}$$

It also acts on functions $f(\mathfrak{x}) \in \mathbf{S}$ as

$$[\theta f](\mathfrak{x}) = \int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{x}'\ \theta(\mathfrak{x}, \mathfrak{x}') f(\mathfrak{x}'); \qquad \theta(\mathfrak{x}, \mathfrak{x}') = \int_{\widehat{\mathbb{Z}}} ds \int_{\widehat{\mathbb{Z}}} ds' \chi(s'\mathfrak{x}' - s\mathfrak{x}) \theta(s, s'), \tag{68}$$

Its trace is defined as:

$$\mathrm{tr}\theta = \int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{x}\ \theta(\mathfrak{x}, \mathfrak{x}) = \int_{\widehat{\mathbb{Z}}} ds\ \theta(s, s). \tag{69}$$

Theorem VII.1. *For any trace class operator θ*

$$\frac{1}{2} \int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{a} \ \int_{\widehat{\mathbb{Z}}} db\ D(\mathfrak{a}, b, 0)\ \theta\ [D(\mathfrak{a}, b, 0)]^\dagger = \mathbf{1} \mathrm{tr}\theta. \tag{70}$$

Proof. Using Eq.(39), we act with $D(\mathfrak{a}, b, 0)\ \theta\ [D(\mathfrak{a}, b, 0)]^\dagger$ on an arbitrary $f(\mathfrak{x}) \in \mathbf{S}$ and we get

$$\left[D(\mathfrak{a}, b, 0)\ \theta\ [D(\mathfrak{a}, b, 0)]^\dagger f\right](\mathfrak{x}) = \int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{x}'\ \chi(2\mathfrak{a}b - \mathfrak{x}b + \mathfrak{x}'b)\ \theta(\mathfrak{x} - 2\mathfrak{a}, \mathfrak{x}')\ f(\mathfrak{x}' + 2\mathfrak{a}) \tag{71}$$

Therefore for an arbitrary $g(\mathfrak{x}) \in \mathbf{S}$ we get

$$\begin{aligned} &\int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{a} \ \int_{\widehat{\mathbb{Z}}} db\ \left(g, D(\mathfrak{a}, b, 0)\ \theta\ [D(\mathfrak{a}, b, 0)]^\dagger f\right) \\ &= \int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{a} \ \int_{\widehat{\mathbb{Z}}} db\ \int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{x} \int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{x}'\ \overline{g(\mathfrak{x})}\ \chi(2\mathfrak{a}b - \mathfrak{x}b + \mathfrak{x}'b)\ \theta(\mathfrak{x} - 2\mathfrak{a}, \mathfrak{x}')\ f(\mathfrak{x}' + 2\mathfrak{a}) \end{aligned} \tag{72}$$

We then use the relation

$$\int_{\widehat{\mathbb{Z}}} db\, \chi(2\mathfrak{a}b - \mathfrak{x}b + \mathfrak{x}'b) = \Delta(2\mathfrak{a} - \mathfrak{x} + \mathfrak{x}') \tag{73}$$

and taking into account Eqs.(27),(32) we get

$$\int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{a} \int_{\widehat{\mathbb{Z}}} db\, \left(g, D(\mathfrak{a}, b, 0)\,\theta\,[D(\mathfrak{a}, b, 0)]^\dagger f\right) = 2\int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{x}\, \overline{g(\mathfrak{x})}\, f(\mathfrak{x}) \int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{x}'\, \theta(\mathfrak{x}', \mathfrak{x}') \tag{74}$$

The right hand side is $(g, f)\mathrm{tr}\theta$ and this completes the proof. □

The set of coherent states consists of the states

$$f_{\mathrm{coh}}(\mathfrak{x}|\mathfrak{a}, b) \equiv [D(\mathfrak{a}, b, \mathfrak{c})f](\mathfrak{x}); \qquad \mathfrak{a} \in \mathbb{Q}/\mathbb{Z}; \qquad b \in \widehat{\mathbb{Z}} \tag{75}$$

where $f(\mathfrak{x})$ is an arbitrary but fixed state in $\mathbf{S}$ ('fiducial vector') which for convinience we normalize so that $(f, f) = 1$. The coherent states have the 'resolution of the identity' property:

$$\frac{1}{2}\int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{a} \int_{\widehat{\mathbb{Z}}} db\, f_{\mathrm{coh}}(\mathfrak{x}|\mathfrak{a}, b)\, \overline{f_{\mathrm{coh}}(\mathfrak{x}'|\mathfrak{a}, b)} = \Delta(\mathfrak{x} - \mathfrak{x}'). \tag{76}$$

We prove this using the above theorem with $\theta(\mathfrak{x}, \mathfrak{x}') = f(\mathfrak{x})\overline{f(\mathfrak{x}')}$. The $\Delta(\mathfrak{x} - \mathfrak{x}')$ is the kernel of the identity operator $\mathbf{1}$.

Theorem VII.2. *A trace class operator θ can be expanded in terms of displacement operators, as*

$$\theta = \frac{1}{2}\int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{a} \int_{\widehat{\mathbb{Z}}} db\, D(\mathfrak{a}, b, 0)\mathrm{tr}[\theta D(-\mathfrak{a}, -b, 0)]; \qquad \mathfrak{a} \in \mathbb{Q}/\mathbb{Z}; \qquad b \in \widehat{\mathbb{Z}} \tag{77}$$

Proof. We first act with the operator $\theta D(-\mathfrak{a}, -b, 0)$ on a function $f(\mathfrak{x}) \in \mathbf{S}$, and we get

$$\theta D(-\mathfrak{a}, -b, 0)f(\mathfrak{x}) = \int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{x}' \int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{x}''\, \theta(\mathfrak{x}, \mathfrak{x}')\chi\left(\mathfrak{a}b + \mathfrak{x}'b\right)\Delta(\mathfrak{x}'' - \mathfrak{x}' - 2\mathfrak{a})f(\mathfrak{x}''). \tag{78}$$

Therefore according to Eq.(69), its trace is

$$\begin{aligned} \mathrm{tr}[\theta D(-\mathfrak{a}, -b, 0)] &= \int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{x} \int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{x}'\, \theta(\mathfrak{x}, \mathfrak{x}')\chi\left(\mathfrak{a}b + \mathfrak{x}'b\right)\Delta(\mathfrak{x} - \mathfrak{x}' - 2\mathfrak{a}) \\ &= \int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{x}'\, \theta(\mathfrak{x}' + 2\mathfrak{a}, \mathfrak{x}')\chi\left(\mathfrak{a}b + \mathfrak{x}'b\right). \end{aligned} \tag{79}$$

We now act with the operator on the right hand side of Eq.(77) on a function $f(\mathfrak{x}) \in \mathbf{S}$, and we get

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{a} \int_{\widehat{\mathbb{Z}}} db \int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{x}'\, \theta(\mathfrak{x}' + 2\mathfrak{a}, \mathfrak{x}')\chi\left(\mathfrak{a}b + \mathfrak{x}'b\right)\ \chi\left(\mathfrak{a}b - \mathfrak{x}b\right) f(\mathfrak{x} - 2\mathfrak{a}) \\ &= \frac{1}{2}\int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{a} \int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{x}'\, \theta(\mathfrak{x}' + 2\mathfrak{a}, \mathfrak{x}')\, \delta(2\mathfrak{a} + \mathfrak{x}' - \mathfrak{x})f(\mathfrak{x} - 2\mathfrak{a}) \\ &= \frac{1}{2}\int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{a}\, \theta(\mathfrak{x}, \mathfrak{x} - 2\mathfrak{a})f(\mathfrak{x} - 2\mathfrak{a}) = \int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{x}'\, \theta(\mathfrak{x}, \mathfrak{x}')f(\mathfrak{x}') \end{aligned} \tag{80}$$

Eq.(27) has been used in the change of variables $\mathfrak{x}' = \mathfrak{x} - 2\mathfrak{a}$. The result shows that the right hand side of Eq.(77) is equal to the operator θ. □

VIII. PARITY OPERATORS

The parity operator around the origin $P(0,0)$ acts on the functions $F(s) \in \mathbf{S}$, as follows:

$$P(0,0)F(s) = F(-s) \tag{81}$$

The parity operator around the $(\mathfrak{a}, b)$ where $\mathfrak{a} \in \mathbb{Q}/(\frac{1}{2}\mathbb{Z})$ and $b \in \widehat{\mathbb{Z}}$ is given by

$$P(\mathfrak{a}, b) = D(\mathfrak{a}, b, \mathfrak{c})\ P(0,0)\ [D(\mathfrak{a}, b, \mathfrak{c})]^\dagger \tag{82}$$

Here $\mathfrak{a} \in \mathbb{Q}/(\frac{1}{2}\mathbb{Z})$ because we can easily show (using Eq.(86) proved below) that

$$P\left(\mathfrak{a} + \frac{1}{2}, b\right) = P(\mathfrak{a}, b) \tag{83}$$

Proposition VIII.1.

(1) $P(\mathfrak{a}, b)$ *acts on functions* $F(s) \in \mathbf{S}$ *as follows:*

$$[P(\mathfrak{a}, b)F](s) = \chi[-4\mathfrak{a}(b+s)]F(-s-2b) \tag{84}$$

It also acts on functions $f(\mathfrak{x}) \in \mathbf{S}$ *as follows:*

$$[P(\mathfrak{a}, b)f](\mathfrak{x}) = \chi[2b(2\mathfrak{a} + \mathfrak{x})]f(-\mathfrak{x} - 4\mathfrak{a}) \tag{85}$$

(2)

$$P(\mathfrak{a}, b) = D(2\mathfrak{a}, 2b, 0)\ P(0,0) = P(0,0)\ [D(2\mathfrak{a}, 2b, 0)]^\dagger; \quad [P(\mathfrak{a}, b)]^2 = \mathbf{1} \tag{86}$$

(3)

$$P(\mathfrak{a}, b)P(\mathfrak{a}', b') = D(2\mathfrak{a} - 2\mathfrak{a}', 2b - 2b', 4b\mathfrak{a}' - 4b'\mathfrak{a}) \tag{87}$$

(4)

$$P(\mathfrak{a}, b) = \int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{a}' \int_{\mathbb{Z}} db'\ D(\mathfrak{a}', b', 0)\chi(2\mathfrak{a}'b - 2\mathfrak{a}b') \tag{88}$$

Proof.

(1) Combining Eq.(82) with Eqs(38),(39), we prove Eqs.(84), (85).

(2) Acting with either $D(2\mathfrak{a}, 2b, 0)\ P(0,0)$ or $P(0,0)\ [D(2\mathfrak{a}, 2b, 0)]^\dagger$ on an arbitrary function $F(s) \in \mathbf{S}$, we get the result in Eq.(84). This proves the first of Eqs.(86). The second one is straightforward.

(3) This is easily proved using the $P(\mathfrak{a}, b) = D(2\mathfrak{a}, 2b, 0)\ P(0,0)$ and $P(\mathfrak{a}', b') = P(0,0)\ D(-2\mathfrak{a}', -2b', 0)$.

(4) Acting with the right hand side of Eq.(88) on an arbitrary function $f(\mathfrak{x}) \in \mathbf{S}$, we get

$$\int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{a}' \left[\chi(2\mathfrak{a}'b)f(\mathfrak{x} - 2\mathfrak{a}') \int_{\mathbb{Z}} db'\ \chi[b'(\mathfrak{a}' - \mathfrak{x} - 2\mathfrak{a})\right] = \chi[2b(2\mathfrak{a} + \mathfrak{x})]\ f(-\mathfrak{x} - 4\mathfrak{a}) \tag{89}$$

Taking into acount Eq.(85), we see that this proves Eq.(88).

□

Theorem VIII.2. *A trace class operator θ can be expanded in terms of parity operators, as*

$$\theta = \frac{1}{8}\int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{a} \int_{\widehat{\mathbb{Z}}} db \; P(\mathfrak{a}, b)\; \mathrm{tr}[\theta P(\mathfrak{a}, b)]; \qquad \mathfrak{a} \in \mathbb{Q}/\mathbb{Z}; \qquad b \in \widehat{\mathbb{Z}} \tag{90}$$

Proof. We act with the operator $\theta P(\mathfrak{a}, b)$ on a function $f(\mathfrak{x}) \in \mathbf{S}$, and we get

$$\theta P(\mathfrak{a}, b) f(\mathfrak{x}) = \int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{x}' \int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{x}'' \; \theta(\mathfrak{x}, \mathfrak{x}')\chi\left(4\mathfrak{a}b + 2\mathfrak{x}'b\right) \Delta(\mathfrak{x}'' + \mathfrak{x}' + 4\mathfrak{a}) f(\mathfrak{x}''). \tag{91}$$

Therefore according to Eq.(69), its trace is

$$\begin{aligned} \mathrm{tr}[\theta P(\mathfrak{a}, b)] &= \int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{x} \int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{x}' \; \theta(\mathfrak{x}, \mathfrak{x}')\chi\left(4\mathfrak{a}b + 2\mathfrak{x}'b\right) \Delta(\mathfrak{x} + \mathfrak{x}' + 4\mathfrak{a}) \\ &= \int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{x}\; \theta(\mathfrak{x}, -\mathfrak{x} - 4\mathfrak{a})\chi(-4\mathfrak{a}b - 2\mathfrak{x}b). \end{aligned} \tag{92}$$

We now act with the operator on the right hand side of Eq.(90) on a function $f(\mathfrak{x}) \in \mathbf{S}$, and we get

$$\begin{aligned} &\frac{1}{8}\int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{a} \int_{\widehat{\mathbb{Z}}} db \int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{x}' \; \theta(\mathfrak{x}', -\mathfrak{x}' - 4\mathfrak{a})\chi(-4\mathfrak{a}b - 2\mathfrak{x}'b)\chi(4\mathfrak{a}b + 2\mathfrak{x}b) f(-\mathfrak{x} - 4\mathfrak{a}) \\ &= \frac{1}{8}\int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{a} \int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{x}' \; \theta(\mathfrak{x}', -\mathfrak{x}' - 4\mathfrak{a})\Delta(2\mathfrak{x} - 2\mathfrak{x}') f(-\mathfrak{x} - 4\mathfrak{a}) \\ &= \frac{1}{4}\int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{a}\; \theta(\mathfrak{x}, -\mathfrak{x} - 4\mathfrak{a}) f(-\mathfrak{x} - 4\mathfrak{a}) = \int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{x}' \theta(\mathfrak{x}, \mathfrak{x}') f(\mathfrak{x}') \end{aligned} \tag{93}$$

Eqs.(27),(32) have been taken into account, in changes of variables. This proves the theorem.

□

IX. WIGNER AND WEYL FUNCTIONS

Given a pair of functions $g(\mathfrak{x}), f(\mathfrak{x}) \in \mathbf{S}$, their Wigner function $W(\mathfrak{a}, b; g, f)$ and their Weyl (or ambiguity) function $\widetilde{W}(\mathfrak{a}, b; g, f)$ are defined as

$$W(\mathfrak{a}, b; g, f) = (g, P(\mathfrak{a}, b) f); \qquad \widetilde{W}(\mathfrak{a}, b; g, f) = (g, D(\mathfrak{a}, b, 0) f) \tag{94}$$

If $g = f$, they are the auto-Wigner and auto-Weyl functions of f.

Proposition IX.1.

$$W(\mathfrak{a}, b; g, f) = \int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{a}' \int_{\mathbb{Z}} db' \; \widetilde{W}(\mathfrak{a}', b'; g, f)\chi(2\mathfrak{a}'b - 2\mathfrak{a}b') \tag{95}$$

Proof. □

This is a direct consequence of Eq.(88).

Proposition IX.2. *The 'marginal properties' of the Wigner function are as follows:*

$$\int_{\widehat{\mathbb{Z}}} db\, W(\mathfrak{a}, b; g, f) = 2\overline{g(-2\mathfrak{a})}\, f(-2\mathfrak{a})$$
$$\int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{a}\, W(\mathfrak{a}, b; g, f) = 4\, \overline{\widetilde{g}(-b)}\, \widetilde{f}(-b). \tag{96}$$

Here tilde denotes the Fourier transform of Eq.(26).

Proof. We use Eq.(85) to prove that

$$\int_{\widehat{\mathbb{Z}}} db\, W(\mathfrak{a}, b; g, f) = \int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{x} \int_{\widehat{\mathbb{Z}}} db\, \overline{g(\mathfrak{x})} \chi\left(4\mathfrak{a}b + 2\mathfrak{x}b\right) f(-\mathfrak{x} - 4\mathfrak{a})$$
$$= \int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{x}\, \overline{g(\mathfrak{x})}\, \Delta(4\mathfrak{a} + 2\mathfrak{x})\, f(-\mathfrak{x} - 4\mathfrak{a}) = 2\overline{g(-2\mathfrak{a})}\, f(-2\mathfrak{a}) \tag{97}$$

Here we changed variables taking into acccount Eq.(27), and this gave the factor 2.

We also use Eq.(85) to prove that

$$\int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{a}\, W(\mathfrak{a}, b; g, f) = \int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{x} \int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{a}\, \overline{g(\mathfrak{x})} \chi\left(4\mathfrak{a}b + 2\mathfrak{x}b\right) f(-\mathfrak{x} - 4\mathfrak{a}) \tag{98}$$

We then change variables from $(\mathfrak{x}, \mathfrak{a})$ to $(\mathfrak{x}, \mathfrak{x}' = -\mathfrak{x} - 4\mathfrak{a})$, and taking into account Eq.(27), we get

$$\int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{a}\, W(\mathfrak{a}, b; g, f) = 4 \int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{x}\, \overline{g(\mathfrak{x})} \chi\left(b\mathfrak{x}\right) \int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{x}'\, \chi\left(-b\mathfrak{x}'\right) f(\mathfrak{x}') = 4\, \overline{\widetilde{g}(-b)}\, \widetilde{f}(-b). \tag{99}$$

□

In the p-adic representation 2^{-1} is a p-adic integer for all $p \neq 2$. Therefore if $b \in \widehat{\mathbb{Z}}_{\rm odd}$, then $2^{-1}b \notin \widehat{\mathbb{Z}}$ and if $b \in \widehat{\mathbb{Z}}_{\rm even}$, then $2^{-1}b \in \widehat{\mathbb{Z}}$.

For $b = (b_2, b_3, ..., b_p, ...) \in \widehat{\mathbb{Z}}_{\rm odd}$, and for factorizable functions $f(\mathfrak{x}) = \prod f_p(\mathfrak{x}_p)$ as in Eq.(23), we define the

$$\check{f}(2^{-1}b) = \int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{x} \chi(2^{-1}b\mathfrak{x}) f(\mathfrak{x}) = \check{f}_2(2^{-1}b_2) \prod_{p \neq 2} \widetilde{f}_p(2^{-1}b_p) \tag{100}$$

where

$$\check{f}_2(2^{-1}b_2) = \int_{\mathbb{Q}_2/\mathbb{Z}_2} d\mathfrak{x}_2 \chi_2(2^{-1}b_2\mathfrak{x}_2) f_2(\mathfrak{x}_2) \tag{101}$$

Since $2^{-1}b_2 \notin \mathbb{Z}_2$, the $\check{f}_2(2^{-1}b_2)$ is not the Fourier transform of $f_2(\mathfrak{x}_2)$. Therefore $\check{f}(2^{-1}b)$ is not the Fourier transform of $f(\mathfrak{x})$ (although for $p \neq 2$ the $\widetilde{f}_p(2^{-1}b_p)$ are the Fourier transforms of $f_p(\mathfrak{x}_p)$).

Similar comment can be made for finite linear combinations of factorizable functions.

Proposition IX.3. *The 'marginal properties' of the Weyl function are as follows:*

(1)

$$\int_{\widehat{\mathbb{Z}}} db\, \widetilde{W}(\mathfrak{a}, b; g, f) = \overline{g(\mathfrak{a})}\, f(-\mathfrak{a}), \tag{102}$$

(2)

$$\int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{a}\, \widetilde{W}(\mathfrak{a},b;g,f) = 2\,\overline{\widetilde{g}(2^{-1}b)}\,\widetilde{f}(-2^{-1}b) \quad \text{if}\ \ b\in\widehat{\mathbb{Z}}_{\rm even}$$
$$\int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{a}\, \widetilde{W}(\mathfrak{a},b;g,f) = 2\,\overline{\check{g}(2^{-1}b)}\,\check{f}(-2^{-1}b) \quad \text{if}\ \ b\in\widehat{\mathbb{Z}}_{\rm odd} \tag{103}$$

where tilde denotes the Fourier transform of Eq.(26) and $\check{f}$ has been defined in Eq.(100).

Proof.

(1) We use Eq.(39) to prove that

$$\begin{aligned}\int_{\widehat{\mathbb{Z}}} db\, \widetilde{W}(\mathfrak{a},b;g,f) &= \int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{x} \int_{\widehat{\mathbb{Z}}} db\, \overline{g(\mathfrak{x})}\chi\left(\mathfrak{a}b-\mathfrak{x}b\right) f(\mathfrak{x}-2\mathfrak{a})\\ &= \int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{x}\, \overline{g(\mathfrak{x})}\Delta(\mathfrak{a}-\mathfrak{x})f(\mathfrak{x}-2\mathfrak{a}) = \overline{g(\mathfrak{a})}\, f(-\mathfrak{a})\end{aligned} \tag{104}$$

(2) We use Eq.(39) to prove that

$$\int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{a}\, \widetilde{W}(\mathfrak{a},b;g,f) = \int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{x} \int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{a}\, \overline{g(\mathfrak{x})}\chi\left(\mathfrak{a}b-\mathfrak{x}b\right) f(\mathfrak{x}-2\mathfrak{a}) \tag{105}$$

We then change variables from $(\mathfrak{x},\mathfrak{a})$ to $(\mathfrak{x},\mathfrak{x}'=\mathfrak{x}-2\mathfrak{a})$ and taking into account Eq.(27), we get

$$\int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{a}\, \widetilde{W}(\mathfrak{a},b;g,f) = 2\int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{x}\, \overline{g(\mathfrak{x})}\chi\left(-\frac{b}{2}\mathfrak{x}\right) \int_{\mathbb{Q}/\mathbb{Z}} d\mathfrak{x}'\, \chi\left(-\frac{b}{2}\mathfrak{x}'\right) f(\mathfrak{x}'). \tag{106}$$

This in conjuction with the remark above, prove Eq.(103).

□

X. DISCUSSION

We have studied quantum systems with positions in $\mathbb{Q}/\mathbb{Z}$ and momenta in $\widehat{\mathbb{Z}}$. We have presented a phase space formalism in this context. In particular we have given several properties for the displacement and parity operators, and also for the Wigner and Weyl functions. A wide class of Hamiltonians which can be used in time evolution calculations, has been studied. From a physical point of view, we have shown that a finite number of coupled finite quantum systems can be embedded into this formalism. The formalism is at the 'edge' of the formalisms for finite systems with variables in $\mathbb{Z}(d)$.

Other finite quantum systems are systems where position and momentum takes values in the Galois field $GF(p^n)$ (for a review see [39]). Taking the direct and inverse limit in these systems we get quantum mechanics in infinite Galois fields and their Pontryagin dual groups which are profinite (for the mathematical aspects of such models see [40, 41]). Finally we mention quantum mechanics on adeles studied in [42–44].

The work combines algebraic number theory with quantum mechanics.

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TABLE I: Some groups and their Pontryagin dual groups

group	Pontryagin dual group
$\mathbb{Z}(d)$	$\mathbb{Z}(d)$
$\mathbb{Q}_p/\mathbb{Z}_p$	$\mathbb{Z}_p$
$\mathbb{Q}/\mathbb{Z} = \prod \mathbb{Q}_p/\mathbb{Z}_p$	$\widehat{\mathbb{Z}} = \prod \mathbb{Z}_p$