Quantum probabilities as non-additive probabilities

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- Kolmogorov (additive) probs: $q(A)$
  Boolean algebra of powerset 2^{Ω} of a set Ω

  $$\delta(A,B) = q(A \vee B) - q(A) - q(B) + q(A \wedge B)$$
  $$\delta(A,B) = 0; \quad \vee = \cup; \quad \wedge = \cap; \quad A,B \subseteq \Omega$$

  additivity

  $$A \cap B = \emptyset \;\; \rightarrow \;\; q(A \vee B) = q(A) + q(B)$$

  Boole's inequality

  $$q(A \vee B) \leq q(A) + q(B)$$

  additivity

- non-addditive probs:
  the whole is greater than the sum of its parts
  added value in coalitions/aggregations
  eg coalition of two political parties
  Keynes, Koopman, ...

- within non-addditive probs:
  Dempster-Shafer theory: lower and upper probs
  Artifical Intelligence, Operational Research, Economics, etc
  Here $\delta(A,B) \neq 0$

- system: variables in $\mathbb{Z}(d)$; Hilbert space $H(d)$
  two different lattices: analogue of $\delta(A,B) \neq 0$
  quantum probs are **not** Kolmogorov probs
  - Birkhoff-von Neumann modular orthocomplemented lattice of subspaces $\mathcal{L}[H(d)]$
    non-distributive (modularity=weak distributivity)
    non-commutativity $\rightarrow \mathfrak{D}(H_1, H_2) \neq 0$
  - lattice $\Lambda[H(d)]$ of subsystems
    subsystem: variables in subgroup of $\mathbb{Z}(d)$ i.e in $\mathbb{Z}(e)$, with $e|d$ (subsystem $\neq$ subspace)
    distributive; projectors commute
    Heyting alg (not Boolean) $\rightarrow \Delta(m_1, m_2) \neq 0$

  sublattices Boolean algebras: $\delta(A,B) = 0$
  quantum probs=Kolmogorov probs in 'islands'

  full lattice: non-additive probs, $\delta(A,B) \neq 0$
  Dempster-Shafer theory: lower and upper probs

- motivation for studying the second lattice:
  1. algebra studies structure and substructures
  2. is lack of distributivity responsible for non-additive probs (merits of quantum computation) ?

Dempster-Shafer theory (classical context): lower and upper probabilities

- Kolmogorov probs in set X
  multivalued map $X \rightarrow \Omega$
  probs $\rightarrow$ lower and upper probs (also called belief, plausibility)

- student projects assessed by many profs
  many marks same project (multivaluedness)

S_1	60	65	72	
S_2	70	72		
S_3	61	65	68	
S_4	50	55	58	62

prob random student marks in $A = [70, 100]$

$$\ell(A) = \frac{1}{4}; \quad u(A) = \frac{1}{2}$$

lower prob=belong to A
upper=lower + don't know=don't belong to $\overline{A}$

lower prob: student S_2
don't know: student S_1
belong to $\overline{A}$: students S_3, S_4
don't belong to $\overline{A}$: : students S_1, S_2

due to multivaluedness
don't belong to $\overline{A} \neq$ **belong to** A
(link later to Heyting logic)

- lower probs

$$\ell(A \cup B) - \ell(A) - \ell(B) + \ell(A \cap B) \geq 0$$
$$B = \overline{A} \;\; \rightarrow \;\; \ell(\overline{A}) + \ell(A) \leq 1$$

upper probs

$$u(A) = 1 - \ell(\overline{A}) \geq \ell(A)$$
$$u(A \cup B) - u(A) - u(B) + u(A \cap B) \leq 0$$

Kolmogorov probs

$$q(A \cup B) - q(A) - q(B) + q(A \cap B) = 0$$
$$B = \overline{A} \;\; \rightarrow \;\; q(A) + q(\overline{A}) = 1$$

- Kolmogorov and upper probs obey
Boole's inequality

$$q(A \cup B) \leq q(A) + q(B)$$
$$u(A \cup B) \leq u(A) + u(B)$$

Lower probs do not obey it
used in proof of Bell inequalities

- additivity

$$A \cap B = \emptyset \;\; \rightarrow \;\; q(A \cup B) = q(A) + q(B)$$

important in integration
new integration concept: Choquet integrals

lower probabilities $\ell(A)$
$A \subseteq B \quad \rightarrow \quad \ell(A) \leq \ell(B)$
$\ell(\emptyset) = 0; \quad \ell(\Omega) = 1$
$\ell(A \cup B) - \ell(A) - \ell(B) + \ell(A \cap B) \geq 0$
$\ell(\overline{A}) + \ell(A) \leq 1$
$\ell(A) + \ell(B) - \ell(A \cup B)$ might be negative

upper probabilities $u(A) = 1 - \ell(\overline{A})$
$A \subseteq B \quad \rightarrow \quad u(A) \leq u(B)$
$u(\emptyset) = 0; \quad u(\Omega) = 1$
$u(A \cup B) - u(A) - u(B) + u(A \cap B) \leq 0$
$u(\overline{A}) + u(A) \geq 1$
$u(A) + u(B) - u(A \cup B) \geq 0$
Boole's inequality

Kolmogorov probabilities $q(A)$
$A \subseteq B \quad \rightarrow \quad q(A) \leq q(B)$
$q(\emptyset) = 0; \quad q(\Omega) = 1$
$q(A \cup B) - q(A) - q(B) + q(A \cap B) = 0$
$q(\overline{A}) + q(A) = 1$
$q(A) + q(B) - q(A \cup B) \geq 0$ Boole's inequality

Birkhoff-von Neumann modular orthocomplemented lattice of subspaces $\mathcal{L}[H(d)]$

- H_1, H_2 are subspaces of $H(d)$

  $H_1 \wedge H_2 = H_1 \cap H_2$; $\quad H_1 \vee H_2 = \mathrm{span}(H_1 \cup H_2)$

  $\mathfrak{P}(H_i)$ projector to H_i

- **modular, not distributive**
  distributivity:
  $H_1 \vee (H_2 \wedge H_3) = (H_1 \vee H_2) \wedge (H_1 \vee H_3)$
  modularity= weak distributivity
  $H_1 \prec H_2 \ \rightarrow \ H_1 \vee (H_2 \wedge H_3) = (H_1 \vee H_2) \wedge H_3$

- For Kolmogorov probs $\delta(A, B) = 0$

  $$\delta(A, B) = q(A \vee B) - q(A) - q(B) + q(A \wedge B)$$

  In analogy, define the operator

  $$\mathfrak{D}(H_1, H_2) = \mathfrak{P}(H_1 \vee H_2) + \mathfrak{P}(H_1 \wedge H_2) - \mathfrak{P}(H_1) - \mathfrak{P}(H_2)$$

  and prove that

  $$[\mathfrak{P}(H_1), \mathfrak{P}(H_2)] = \mathfrak{D}(H_1, H_2)[\mathfrak{P}(H_1) - \mathfrak{P}(H_2)]$$

  $[\mathfrak{P}(H_1), \mathfrak{P}(H_2)]$ measures non-commutativity
  $\mathfrak{D}(H_1, H_2)$ measures non-additivity of probs
  non-commutativity $\leftrightarrow$non-additivity of probs

- $p(H_i|\rho) = \mathrm{Tr}[\rho\mathfrak{P}(H_i)]$
  $\mathrm{Tr}[\rho\mathfrak{D}(H_1, H_2)] \geq 0$: $p(H_1|\rho), p(H_2|\rho)$ lower probs
  $\mathrm{Tr}[\rho\mathfrak{D}(H_1, H_2)] \leq 0$: $p(H_1|\rho), p(H_2|\rho)$ upper probs

- Dempster multivaluedness:
  classical $\rightarrow$ quantum
  classical AB $\rightarrow$ $\hat{A}\hat{B}$ or $\hat{B}\hat{A}$

- properties of $\mathfrak{D}(H_1, H_2)$ related to properties of modular lattice (projective geometry)
  $\mathrm{Tr}[\mathfrak{D}(H_1, H_2)] = 0$,....
  for $\rho = \frac{1}{d}\mathbf{1}$ (infinite temp): Kolmogorov probs

- Boolean subalgebras in $\mathcal{L}[H(d)]$ (example below)
  $[\mathfrak{P}(H_1), \mathfrak{P}(H_2)] = \mathfrak{D}(H_1, H_2) = 0$
  quantum probs=Kolmogorov probs

  within the full lattice $\mathfrak{D}(H_1, H_2) \neq 0$
  quantum probs=Dempster-Shafer probs

  quantum probs=Kolmogorov probs in ‘islands’ within the full lattice
  In the full lattice: quantum probs=Dempster-Shafer probs

- example of Boolean subalgebra in $\mathcal{L}[H(4)]$ (for two spin 1/2 particles)

$$B_A = \{\mathcal{I}, \mathcal{O}, H_{iA}, H_{iA}^{\perp} \mid i = 1, ..., 7\}$$

closed with respect to $\vee, \wedge$
all spaces commute

$H_{1A} = \left\{a(1,1,1,1)^T\right\}$
$H_{1A}^{\perp} = \left\{(a,b,c,-a-b-c)^T\right\}$
$H_{2A} = \left\{a(1,-1,1,-1)^T\right\}$
$H_{2A}^{\perp} = \left\{(a,b,c,a-b+c)^T\right\}$
$H_{3A} = \left\{a(1,1,-1,-1)^T\right\}$
$H_{3A}^{\perp} = \left\{(a,b,c,a+b-c)^T\right\}$
$H_{4A} = \left\{a(1,-1,-1,1)^T\right\}$
$H_{4A}^{\perp} = \left\{(a,b,c,-a+b+c)^T\right\}$
$H_{5A} = \left\{(a,b,a,b)^T\right\}$
$H_{5A}^{\perp} = \left\{(a,b,-a,-b)^T\right\}$
$H_{6A} = \left\{(a,a,b,b)^T\right\}$
$H_{6A}^{\perp} = \left\{(a,-a,b,-b)^T\right\}$
$H_{7A} = \left\{(a,b,b,a)^T\right\}$ $H_{7A}^{\perp} = \left\{(a,b,-b,-a)^T\right\}$

- with a unitary transformation we get another Boolean subalgebra of $\mathcal{L}[H(4)]$

- implications for Bell inequalities:
  proof based on Boole's inequality
  not valid for Dempster-Shafer probs
  violated in experiments
  supports adoption of Dempster-Shafer probs

- Bell inequality (CHSH):
  if $H_1 \wedge ... \wedge H_n = \mathcal{O}$ then

$$\sum_{i=1}^{n} p(H_i|\rho) \leq n-1$$

Proof:
$(H_1 \wedge ... \wedge H_n)^\perp = (H_1^\perp \vee ... \vee H_n^\perp) = \mathcal{I}$
$p(H_1^\perp \vee ... \vee H_n^\perp|\rho) = 1$
with Boole's inequality
$\sum p(H_i^\perp|\rho) \geq 1 \;\rightarrow\; \sum[1 - p(H_i|\rho)] \geq 1$

with expectation values

$\langle H_i; \rho\rangle = p(H_i|\rho) - [1 - p(H_i|\rho)] = 2p(H_i|\rho) - 1$

it becomes

$$\sum_{i=1}^{n} \langle H_i; \rho\rangle \leq n-2$$

The lattice of subsystems: Heyting algebra

first the lattice $\Lambda(d)$ of divisors of d
second the lattice of subsystems $\Lambda[H(d)]$
isomorphic to each other

- $\mathbb{D}(d)$: divisors of d

  $k \vee m = \mathrm{LCM}(k,m); \quad k \wedge m = \mathrm{GCD}(k,m)$

  negation $\neg a$: largest divisor of d, coprime to a
  $\neg a$: complementary 'information' to a

- finite distributive lattice $\mathcal{O} = 1$ and $\mathcal{I} = d$
  it is NOT a Boolean algebra

  Boolean algebra:

  - $\neg a \vee a = \mathcal{I}$ law excluded middle
    something is true or not true
  - two negations = positive statement
    $\neg\neg a = a \quad \leftrightarrow \quad p(A) = 1 - p(\overline{A})$
    fits with Kolmogorov probs

  Heyting algebra: intuitionistic (multivalued) logic

  - $\neg a \vee a \prec \mathcal{I}$
    true/not true/don't know
  - two negations $\succ$ positive statement
    $a \prec \neg\neg a \quad \leftrightarrow \quad \ell(A) \le 1 - \ell(\overline{A}) = u(A)$
    fits with Dempster-Shaffer

    Heyting algebra has Boolean subalgebras

- $\Lambda(18) = \{1, 2, 3, 6, 9, 18\}$
  $\mathcal{O} = 1, \mathcal{I} = 18$
  $\neg\neg 6 = 18$, $6 \vee \neg 6 = 6 \neq \mathcal{I}$ (not Boolean)

Heyting algebra $\Lambda[H(d)]$ of subsystems of $\Sigma(d)$

- quantum system $\Sigma(d)$ with variables in $\mathbb{Z}(d)$
  $|X_d; r\rangle$, $r \in \mathbb{Z}(d)$, basis of position states

- subsystems:
  - **variables in $\Sigma(m)$ in subgroup of the group of variables of $\Sigma(d)$**(therefore $m|d$)
  - embeddings between these systems
    $H(m)$ embedded into $H(d)$ (where $m|d$)

$$|X_m; r\rangle \;\rightarrow\; |X_d; \frac{dr}{m}\rangle$$

    example:
    $\mathbb{Z}(3)$ embedded into $\mathbb{Z}(6)$: 0,2,4
    $H(3)$ embedded into $H(6)$
    superpositions of $|X_6; 0\rangle$, $|X_6; 2\rangle$, $|X_6; 4\rangle$

- Heyting algebra $\Lambda[H(d)]$
  isomorphic to $\Lambda(d)$:divisor $\leftrightarrow$ subsystem
  set of all subsystems of $\Sigma(d)$, i.e. $\Sigma(k)$ with $k|d$

  – $\Sigma(k) \vee \Sigma(m) = \Sigma(k \vee m)$ ($k \vee m = \text{LCM}(k,m)$)
    smallest subsystem of $\Sigma(d)$ which has $\Sigma(k)$, $\Sigma(m)$ as subsystems
    'merger' of $\Sigma(k), \Sigma(m)$
    analogue of LCM for integers

  – $\Sigma(k) \wedge \Sigma(m) = \Sigma(k \wedge m)$ ($k \wedge m = \text{GCD}(k,m)$)
    largest subsystem of both $\Sigma(k)$, $\Sigma(m)$
    analogue of GCD for integers

  – $\neg\Sigma(k) = \Sigma(\neg k)$
    complementary information (they only share lowest state $|X_m; 0\rangle$)

  A. Vourdas, JMP54, 082105 (2013)

- projector to subsystem $\Sigma(m)$ of $\Sigma(d)$:

$$\mathfrak{P}(m) = \sum_{r=0}^{m-1} |X_m; r\rangle\langle X_m; r| = \sum_{r=0}^{m-1} |X_d; \frac{dr}{m}\rangle\langle X_d; \frac{dr}{m}|$$

  projectors to subsystems commute

- $$\begin{aligned}\Delta(m_1,m_2) &= \mathfrak{P}(m_1 \vee m_2) - \mathfrak{P}(m_1) - \mathfrak{P}(m_2)\\ &+ \mathfrak{P}(m_1 \wedge m_2)\end{aligned}$$

  in general non-zero: Dempster-Shafer probs

  if $m_1 | m_2$, i.e. m_1, m_2 in the same chain (Boolean subalgebra) then $\Delta(m_1, m_2) = 0$
  Quantum probs=Kolmogorov probs

- **example:**
  $\Sigma(2)$, $\Sigma(3)$ subsystems of $\Sigma(6)$
  $\mathfrak{P}(2) = |X_6; 0\rangle\langle X_6; 0| + |X_6; 3\rangle\langle X_6; 3|$
  $\mathfrak{P}(3) = |X_6; 0\rangle\langle X_6; 0| + |X_6; 2\rangle\langle X_6; 2| + |X_6; 4\rangle\langle X_6; 4|$
  $\mathfrak{P}(2 \vee 3) = \mathbf{1}$
  $\mathfrak{P}(2 \wedge 3) = |X_6; 0\rangle\langle X_6; 0|$

  $\Delta(2,3) = |X_6; 1\rangle\langle X_6; 1| + |X_6; 5\rangle\langle X_6; 5|$

- 'belongs in $\Sigma(m)$' $\neq$ 'does not belong in $\neg\widetilde{\Sigma}(m)$'
  $\Sigma(2)$ (as subsystem of $\Sigma(6)$)
  superpositions of $|X_6; 0\rangle, |X_6; 3\rangle$

  $\neg\Sigma(2) = \Sigma(\neg 2) = \Sigma(3)$
  $\neg\widetilde{\Sigma}(3)$: superpositions of $|X_6; 2\rangle, |X_6; 4\rangle$

  $|X_6; 3\rangle + |X_6; 5\rangle$ 'does not belong in $\neg\widetilde{\Sigma}(3)$' but it belongs only **partly** in $\Sigma(2)$

Discussion

- For Kolmogorov probs

  $$q(A \vee B) - q(A) - q(B) + q(A \wedge B) = 0$$

  for Dempster-Shafer probs, negative or positive

- system with Hilbert space $H(d)$
  Birkhoff-von Neumann lattice of subspaces $\mathcal{L}[H(d)]$
  non-commutativity $\leftrightarrow$ non-additivity of probs
  $[\mathfrak{P}(H_1), \mathfrak{P}(H_2)] = \mathfrak{D}(H_1, H_2)[\mathfrak{P}(H_1) - \mathfrak{P}(H_2)]$

  $[\mathfrak{P}(H_1), \mathfrak{P}(H_2)] = \mathfrak{D}(H_1, H_2) = 0$ Boolean subalg
  full lattice, $\mathfrak{D}(H_1, H_2) \neq 0$, Dempster-Shafer probs

- lattice $\Lambda[H(d)]$ of subsystems
  projectors commute
  Heyting algebra; not Boolean algebra
  Heyting logic fits Dempster-Shafer probs
  $\Delta(m_1, m_2) \neq 0$

  $\Delta(m_1, m_2) = 0$ only if m_1, m_2 in the same chain
  full lattice, $\Delta(m_1, m_2) \neq 0$, Dempster-Shafer probs

- implication of these ideas for complexity?
  classical computer: Boolean algebra
  quantum computer: modular orthocomplemented lattice with non-additive probs
  distributivity $\rightarrow$ modularity

A. Vourdas, JPA47 345203 (2014)
A. Vourdas, JMP55 082107 (2014)