Analytic representations in the complex plane and unit disc

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Euclidean case: complex plane

- Background formalism
- Gaussian states
- Bargmann representation (growth)
- zeros of Bargmann functions (growth/density of zeros)
- Hadamard's theorem

A. Vourdas, J. Phys. A39 (2006) R65

Background formalism

- Hilbert space: $f(x)$ where $x \in R$

$$(f,g) = \int_{-\infty}^{\infty} dx [f(x)]^* g(x)$$

  x-representation
  normalize $(f,f) = 1$

- orthonormal basis: 'number states'

$$u_N(x) = \pi^{-1/4}\left[2^N N!\right]^{-1/2} H_N(x) \exp\left(-\frac{1}{2}x^2\right)$$

$$f(x) = \sum_N f_N u_N(x); \qquad f_N = \int f(x) u_N(x) dx$$

  N=0,1,2...

- p-representation

$$\tilde{f}(p) = (2\pi)^{-1/2} \int (dx) f(x) e^{-ixp}$$

- position/momentum operators:

$$\begin{array}{l} \mathrm{x-repr:} \quad x; \qquad p = -i\partial_x \\ \mathrm{p-repr:} \quad x = i\partial_p; \qquad p \\ [x,p] = i\mathbf{1} \end{array}$$

also:

$$a = 2^{-1/2}(x + ip) = 2^{-1/2}(x + \partial_x)$$
$$a^\dagger = 2^{-1/2}(x - ip) = 2^{-1/2}(x - \partial_x)$$
$$[a, a^\dagger] = \mathbf{1}$$

$$a u_N(x) = \sqrt{N} u_{N-1}(x)$$
$$a^\dagger u_N(x) = \sqrt{N+1} u_{N+1}(x)$$
$$a^\dagger a u_N(x) = N u_N(x)$$

- Phase space: $x - p$ plane
  function $f(x)$ located around $(< x >, < p >)$
  uncertainty ellipse Δx, Δp, σ_{xp}:

$$< x^M >= \int dx\, x^M |f(x)|^2; \quad < p^M >= \int dp\, p^M |\tilde{f}(p)|^2$$
$$\Delta x = [< x^2 > - < x >^2]^{1/2}; \quad \Delta p$$
$$\sigma_{xp} =< \frac{1}{2}(xp + px) > - < x >< p >$$

- displacements in phase space:

$$D(z) = \exp[z a^\dagger - z^* a]$$
$$D(z) f(x) = f(x - \sqrt{2} z_R) \exp(i\sqrt{2} z_I x - i z_R z_I)$$

average location of function $f(x)$ displaced:

$$(< x >, < p >) \rightarrow (< x > + \sqrt{2} z_R, < p > + \sqrt{2} z_I)$$

Heisenberg-Weyl group

Gaussian states

- For normalized Gaussians

$$(\Delta x \Delta p)^2 - \sigma_{xp}^2 = \frac{1}{4}$$

- **coherent states :**
  Gaussians with $\Delta x = \Delta p = 2^{-1/2}$**;** $\sigma_{xp} = 0$

  'vacuum' (centered at $(0,0)$)

$$u_0(x) = \pi^{-1/4} \exp\left(-\frac{1}{2}x^2\right)$$

  displace it

$$g(x;z) = D(z)u_0(x) = \pi^{-1/4} \exp\left(-\frac{1}{2}x^2 + xz\sqrt{2} - zz_R\right)$$

  centered at $(z_R\sqrt{2}, z_I\sqrt{2})$

- **squeezed states : other Gaussians**

- $g(x;z)$ for all $z \in C$: overcomplete basis
  resolution of the identity

$$\int_C d\mu(z) g(x;z)[g(y;z)]^* = \delta(x-y); \qquad d\mu(z) = \frac{dz_R dz_I}{\pi}$$

arbitrary $f(x)$

$$f(x) = \int_C d\mu(z) g(x;z) F(z); \qquad F(z) = (g(x;z), f(x))$$

- non-orthogonal

$$(g(x;z), g(x;w)) = \exp\left(-\frac{1}{2}|z|^2 - \frac{1}{2}|w|^2 + z^* w\right)$$

- eigenstates of a:

$$a g(x;z) = z g(x;z)$$

- 'temporal stability'
  For a class of Hamiltonians, they evolve into other coherent states within the family

$$\begin{aligned} H &= \omega\left[a^\dagger a - z a^\dagger - z^* a + |z^2|\right] \\ \exp(iHt) g(x;\zeta) &= \exp\left[i|z|^2 \sin\omega t\right] \\ &\times \; g(x; \zeta e^{i\omega t} + z(1 - e^{i\omega t})) \end{aligned}$$

Bargmann representation

- arbitrary function

$$f(x) = \sum_N f_N u_N(x)$$

represented with the Bargmann function

$$f_B(z) = \exp\left[\frac{1}{2}|z|^2\right]([f(x)]^*, g(x;z)) = \sum_{N=0}^{\infty} f_N z^N (N!)^{-1/2}$$

analytic function of z in C.

- growth of entire function $f(z)$:

$$|f| \approx \exp(\sigma R^\rho); \qquad R \to \infty$$

order ρ and type σ
$(\rho, \sigma) < (\rho', \sigma')$ if $\rho < \rho'$ or $\rho = \rho'$ and $\sigma < \sigma'$.

- scalar product

$$(f, g) = \int_C [f_B(z)]^* g_B(z) \exp(-|z|^2) d\mu(z)$$

growth of Bargmann functions smaller than $(\rho = 2, \sigma = 1/2)$.

- In Bargmann representation

$$u_N(x) \to \frac{z^N}{\sqrt{N}}$$
$$a \to \partial_z; \qquad a^\dagger \to z$$
$$a u_N(x) = \sqrt{N} u_{N-1}(x); \quad a^\dagger u_N(x) = \sqrt{N+1} u_{N+1}(x)$$

- coherent states: order $\rho = 1$, $\sigma = |w|$:

$$g(x; w) \to g_B(z; w) = \exp\left[wz - \frac{1}{2}|w|^2\right]$$

special case vacuum

$$g(x; 0) = u_0(x) \to g_B(z; 0) = 1$$

- squeezed states: order $\rho = 2$, $\sigma = |\alpha|/2$

$$f_B(z) = (1 - |\alpha|^2)^{1/4} \exp\left[\frac{1}{2}\alpha z^2 + \beta z + \gamma\right]; \qquad |\alpha| < 1$$

- example with a given order ρ and σ:

$$f(x) = \mathcal{N} \sum_{N=0}^{\infty} f_N u_N(x); \qquad f_N = \frac{e^{i\theta_N} \sigma^{N/\rho} (N!)^{1/2}}{\Gamma(\frac{N}{\rho} + 1)}$$
$$\mathcal{N} = \left[\sum_{N=0}^{\infty} \frac{\sigma^{2N/\rho} N!}{[\Gamma(\frac{N}{\rho} + 1)]^2}\right]^{-\frac{1}{2}}$$

$\{\theta_N\}$ are arbitrary phases.
$\mathcal{N}$ finite, growth less than $(\rho = 2, \sigma = 1/2)$.

- Operator Θ (in the $u_N(x)$ basis Θ_{MN}):

$$\mathcal{K}(z,\zeta^*;\Theta) = \sum_{M,N} \Theta_{MN} z^M (\zeta^*)^N$$

$$(\Theta f)(z) = \int_C d\mu(\zeta) e^{-|\zeta|^2} \mathcal{K}(z,\zeta^*;\mathcal{U}) f(\zeta)$$

examples

$$\begin{aligned} \mathcal{K}(z,\zeta^*;\mathbf{1}) &= \exp(z\zeta^*) \\ \mathcal{K}(z,\zeta^*;a) &= \zeta^* \exp(z\zeta^*) \\ \mathcal{K}(z,\zeta^*;a^\dagger) &= z \exp(z\zeta^*) \end{aligned}$$

$\mathcal{K}(z,\zeta^*;\mathbf{1})$ reproducing kernel,

$$\int_C d\mu(\zeta) e^{-|\zeta|^2 + z\zeta^*} f(\zeta) = f(z)$$

$a, a^\dagger$ differential and integral reprs:

$$\int_C d\mu(\zeta) e^{-|\zeta|^2} \mathcal{K}(z,\zeta^*;a) f(\zeta) = \partial_z f(z)$$
$$\int_C d\mu(\zeta) e^{-|\zeta|^2} \mathcal{K}(z,\zeta^*;a^\dagger) f(\zeta) = z f(z)$$

zeros of Bargmann functions

-
$$f_B(z) = \exp\left[\frac{1}{2}|z|^2\right]([f(x)]^*, g(x;z)) = 0$$

  $[f(x)]^*$ orthogonal to $g(x;z)$

- zeros of analytic functions isolated
  if sequence $\{\zeta_N\}$ converges to (finite) w then corresponding coherent states $g(x;\zeta_N)$ over-complete set.

- Assume zeros $\zeta_1, \zeta_2, \zeta_3...$ are such that

$$0 < |\zeta_1| \leq |\zeta_2| \leq |\zeta_3| \leq \cdot; \qquad \lim_{N\to\infty} |\zeta_N| = \infty$$

  density of zeros (η, δ)

$$n(R) \approx \delta R^\eta; \qquad R \to \infty$$

  η convergence exponent: infimum of λ

$$\sum_{N=1}^{\infty} |\zeta_N|^{-\lambda} < \infty$$

  $(\eta, \delta) < (\eta', \delta')$ if $\eta < \eta'$ or $\eta = \eta'; \delta < \delta'$.

- growth (ρ, σ) of analytic functions related to the density (η, δ) of their zeros

$$\eta \leq \rho; \qquad \sigma\rho \leq \delta$$

- if sequence $\{\zeta_N\}$ has density **smaller/greater** than $(\eta = 2, \delta = 1)$ then corresponding coherent states $g(x; \zeta_N)$ **undercomplete/overcomplete** set.

- **example**:

$$\zeta_N = \left(\frac{N}{\delta_0}\right)^{1/\eta_0} e^{i\theta_N}$$

  θ_N arbitrary phases
  has density $\eta = \eta_0$ and $\delta = \delta_0$
  $\{g(x, \zeta_N)\}$ undercomplete for $\eta_0 = 1.9$ and overcomplete for $\eta_0 = 2.1$

Hadamard's theorem

- Bargmann functions with given zeros $\{\zeta_N\}$
  many with same zeros

$$f_B(z) = z^m \prod_{N=1}^{\infty} E(\zeta_N, p) \exp[Q_q(z)]$$

  m multiplicity of zero at origin
  $Q_q(z)$ is a polynomial of degree q
  E are Weierstrass factors:

$$E(\zeta, 0) = (1 - \frac{z}{\zeta})$$

$$E(\zeta, p) = (1 - \frac{z}{\zeta}) \exp\left[\frac{z}{\zeta} + \frac{z^2}{2\zeta^2} + ... + \frac{z^p}{p\zeta^p}\right]$$

  maximum of (p, q) (genus) does not exceed ρ
  For Bargmann $p, q = 0, 1, 2$.

- $\exp[Q_0(z)]$ (constant) is vacuum
  $\exp[Q_1(z)]$ is coherent state
  $\exp[Q_2(z)]$ is squeezed state

$$z^m \;\rightarrow\; a^{\dagger m}$$

$$\widehat{E}(\zeta_N, 0) = 1 - \frac{a^\dagger}{\zeta_N}$$

$$\widehat{E}(\zeta_N, 1) = \left[1 - \frac{a^\dagger}{\zeta_N}\right] \exp\left[\frac{a^\dagger}{\zeta_N}\right]$$

$$\widehat{E}(\zeta_N, 2) = \left[1 - \frac{a^\dagger}{\zeta_N}\right] \exp\left[\frac{a^\dagger}{\zeta_N} + \frac{a^{\dagger 2}}{2\zeta_N^2}\right]$$

- example: superposition of two Gaussians

  $f(x) = \mathcal{N}[g(x;i) - g(x;-i)] \;\rightarrow\; f_B(z) = 2i\mathcal{N}e^{-1/2}\sin z$

  Hadamard factorization:

  $$\sin z = z\prod_{N=1}^{\infty} E(\zeta_N, 0) = z\prod_{N=1}^{\infty}\left(1 - \frac{z^2}{\pi^2 N^2}\right)$$

  $$\zeta_{2N-1} = -N\pi; \qquad \zeta_{2N} = N\pi$$

  all $g(x; N\pi)$ orthogonal to $f(x)$.

- **von Neumann lattice**
  $\{z_{MN} = S^{1/2}(M + iN)\}$ where M,N integers and S area of lattice cell.
  density is $(\eta = 2, \delta = S/\pi)$
  corresponding $\{g(x; z_{MN})\}$ overcomplete when $S < \pi$; undercomplete when $S > \pi$.

- construct $f(x)$ orthogonal to all $\{g(x; z_{MN})\}$ in a von Neumann lattice with $S = 4$.
  Weierstrass sigma function:

  $$\sigma(z|1, i) = z\prod E(\zeta_{MN}, 2); \qquad \zeta_{MN} = 2(M + iN)$$

  where $(M, N) \neq (0, 0)$ and

  $$E(\zeta_{MN}, 2) = \left[1 - \frac{z}{\zeta_{MN}}\right]\exp\left[\frac{z}{\zeta_{MN}} + \frac{z^2}{2\zeta_{MN}^2}\right]$$

  answer: $f_B(z) = \mathcal{N}\sigma(z|1, i)$

Hyperbolic case: unit disc

1. Bergman space
   - Background formalism
   - $SU(1,1)$ transformations
   - Operators
   - $SU(1,1)$ coherent states
2. Hardy space
   - Background formalism
   - Inner and outer states
   - Zeros of $f(z)$: Blaschke products

Bergman space: background formalism

- Bergman space:
  analytic functions $g(z)$ in D:
  $$\frac{1+\alpha}{\pi}\int_D |g(z)|^p (1-|z|^2)^\alpha dz_R dz_I = 1$$
  where $-1 < \alpha$ and $p > 0$.
  Here $p = 2$ and $\alpha = 2k-2$ with $k = 1/2, 1, 3/2, ..$
  for $k = 1/2$, $\alpha = -1$, as limit

- Hilbert space H_k of $f(z;k)$:
  $$\begin{aligned}(f(z;k), g(z;k)) &= \frac{2k-1}{\pi}\int_D [f(z;k)]^* g(z;k) \\ &\times (1-|z|^2)^{2k} d\mu(z) \\ d\mu(z) &= \frac{dz_R dz_I}{(1-|z|^2)^2}\end{aligned}$$

- orthonormal basis: '$SU(1,1)$ number states
  $$u_N(z;k) = d(N;k) z^N; \qquad d(N;k) = \left[\frac{\Gamma(N+2k)}{\Gamma(N+1)\Gamma(2k)}\right]^{1/2}$$
  where $N = 0, 1, 2, ...$
  $$f(z;k) = \sum_N f_N d(N;k) z^N; \qquad \sum_N |f_N|^2 = 1$$

$SU(1,1)$ transformations

- The operators

  $K_+ = z^2\partial_z + 2kz, \quad K_0 = z\partial_z + k, \quad K_- = \partial_z$

  generators of $SU(1,1)$ group:

  $$[K_0, K_\pm] = \pm K_\pm, \quad [K_-, K_+] = 2K_0$$

  Casimir:

  $$K^2 = K_0^2 - \frac{1}{2}(K_+K_- + K_-K_+) = k(k-1)\mathbf{1}$$

  Here $k = 1/2, 1, 3/2, ...$ discrete series of $SU(1,1)$ representations.

- $$\begin{aligned} K^2 u_N(z;k) &= k(k-1)u_N(z;k), \\ K_0 u_N(z;k) &= (k+N)u_N(z;k), \\ K_- u_N(z;k) &= [N(N+2k-1)]^{1/2} u_{N-1}(z;k) \\ K_+ u_N(z;k) &= [(N+1)(N+2k)]^{1/2} u_{N+1}(z;k), \end{aligned}$$

- $SU(1,1)$ transformations

$$S(r,\theta,\lambda;k) = \exp\left[-\frac{1}{2}re^{-i\theta}K_+ + \frac{1}{2}re^{i\theta}K_-\right]\exp(i\lambda K_0)$$
$$r \geq 0; \quad 0 \leq \theta < 2\pi; \quad 0 \leq \lambda < 2\pi$$

also

$$S(z;k) = S(r,\theta,0;k); \qquad z = -e^{-i\theta}\tanh\frac{r}{2}$$
$$S(z_1;k)\,S(z_2;k) = S(w;k)\exp(-i\phi K_0)$$
$$w = \frac{z_1+z_2}{1+z_1^* z_2}; \quad \phi = 2\arg(1+z_1^* z_2)$$

- the transformations

$$f(z;k) \to S(r,\theta,\lambda;k)f(z;k)$$

implemented with Mobius trans:

$$f(z;k) \to f\left(\frac{az+b}{b^*z+a^*};k\right)(b^*z+a^*)^{-2k}$$

where

$$a = e^{i\lambda/2}\cosh\frac{r}{2}, \quad b = e^{i\theta}\sinh\frac{r}{2}$$

Operators

- operators Θ (in the $u_N(z;k)$ basis Θ_{MN}):

$$\mathcal{K}(z,\zeta^*;\Theta) = \sum_{M,N=0}^{\infty} d(M;k)\, d(N;k)\, \Theta_{MN}\, z^M \zeta^{*N}$$

$$(\Theta f)(z;k) = \frac{2k-1}{\pi} \int_D \mathcal{K}(z,\zeta^*;\Theta) f(\zeta;k)(1-|\zeta|^2)^{2k} d\mu(\zeta)$$

Example:

$$\mathcal{K}(z,\zeta^*;\mathbf{1}) = (1 - z\zeta^*)^{-2k}$$

reproducing kernel,

$$\frac{2k-1}{\pi} \int_D \mathcal{K}(z,\zeta^*;\mathbf{1}) f(\zeta;k)(1-|\zeta|^2)^{2k} d\mu(\zeta) = f(z;k)$$

operators K_0, K_+, K_-

$$\begin{aligned} \mathcal{K}(z,\zeta^*;K_0) &= \frac{2k[-(z\zeta^*)^2 + z\zeta^* + 1]}{(1-z\zeta^*)^{2k}} \\ \mathcal{K}(z,\zeta^*;K_+) &= \frac{2kz}{(1-z\zeta^*)^{2k-1}} \\ \mathcal{K}(z,\zeta^*;K_-) &= \frac{2k\zeta^*}{(1-z\zeta^*)^{2k-1}} \end{aligned}$$

consistent with differential representations

$SU(1,1)$ coherent states

- $SU(1,1)$ coherent states

$$
\begin{aligned}
h(z;w;k) &= S(z;k)u_0(z;k) \\
&= (1-|w|^2)^k \sum_{N=0}^{\infty} d(N;k) w^N u_N(z;k) \\
&= \frac{(1-|w|^2)^k}{(1-zw)^{2k}}; \qquad |z|<1; \quad |w|<1
\end{aligned}
$$

- resolution of the identity

$$\frac{2k-1}{\pi}\int_D h(z;w;k)\,[h(\zeta;w;k)]^* d\mu(w) = (1-z\zeta^*)^{-2k}$$

Arbitrary state $f(z;k)$

$$f(z;k) = \frac{2k-1}{\pi}\int_D h(z;w;k)\;F(w)\;d\mu(w)$$

$$F(w) = \frac{2k-1}{\pi}\int_D f(\zeta;k)\;[h(\zeta;w;k)]^*\;(1-|\zeta|^2)^{2k} d\mu(\zeta)$$

- non-orthogonal states

$$(h(z;w_1;k), h(z;w_2;k)) = \frac{(1-|w_1|^2)^k(1-|w_2|^2)^k}{(1-w_1^* w_2)^{2k}}$$

- for any state $f(z;k)$

  $$f(z;k) = (1-|z|^2)^{-k}(h(\zeta;z^*;k), f(\zeta;k))$$

  $h(z;w^*;k)$ orthogonal to $f(z;k)$ $\leftrightarrow$ w zero of $f(z;k)$

- zeros of analytic functions isolated
  if sequence $\{w_N\}$ converges in the unit disc then corresponding coherent states $h(z;w_N^*,k)$ overcomplete set.

- work needed to connect density of zeros in the unit disc to growth near the unit circle
  Hedenmalm, Korenblum, Zhu

Hardy space: background formalism

- Hardy space:analytic functions $g(z)$ in D:

$$\sup_{0<r<1} \int_0^{2\pi} \frac{d\theta}{2\pi} |g(re^{i\theta})|^p < \infty$$

  here $p = 2$

- scalar product

$$(f, g) = \int_0^{2\pi} \frac{d\theta}{2\pi} \, [f(e^{i\theta})]^* \, g(e^{i\theta})$$

- orthonormal basis $\{z^N\}$, $N = 0, 1, 2...$

$$f(z) = \sum_N f_N z^N$$

  Z-transform (applications: signal processing)

- boundary function

$$f(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta}) = \sum_{N=0}^{\infty} f_N e^{iN\theta}$$

$$\int_0^{2\pi} \frac{d\theta}{2\pi} e^{-iN\theta} f(e^{i\theta}) = f_N; \qquad N \geq 0$$

$$\int_0^{2\pi} \frac{d\theta}{2\pi} e^{-iN\theta} f(e^{i\theta}) = 0; \qquad N < 0$$

- phase states

$$\chi(z;\zeta) = \frac{(1-|\zeta|^2)^{1/2}}{1-\zeta z}; \qquad |\zeta| < 1$$

$P(\theta) = |f(e^{i\theta})|^2$ phase distributions

phase states in the sense

$$P(\theta) = |\chi(e^{i\theta}; re^{i\phi})|^2 = \frac{1-r^2}{1+r^2-2r\cos(\theta+\phi)}$$

For $r \to 1$ we get $P(\theta) \to \delta(\theta+\phi)$.

- for any state $f(z)$

$$f(z) = (1-|z|^2)^{-1/2}(\chi(\zeta; z^*), f(\zeta))$$

$\chi(z; w^*)$ orthogonal to $f(z) \leftrightarrow w$ zero of $f(z)$

Inner and outer states

$f(z)$ factorizes as:

$$f(z) = f_{\text{in}}(z) f_{\text{out}}(z)$$

- outer part of $f(z)$

  $$f_{\text{out}}(z) = \exp[\Phi(z)]$$

  where

  $$\Phi(re^{i\theta}) = \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} [2\mathcal{C}(r, \theta - \phi) - 1] \ln |f(e^{i\phi})|$$

  and

  $$\mathcal{C}(r, \theta) = (1 - re^{i\theta})^{-1}$$

  the Cauchy kernel
  phase distribution $P(\theta) = |f(e^{i\theta})|^2$ defines uniquely outer part of $f(z)$.
  outer part has no zeros

- inner part $f(z)$

  $$f_{\text{in}}(z) = f_{\mathcal{Z}}(z) \exp[-\Phi(z)]$$

  in the interior of the unit disk ($|z| < 1$)

  $$|f_{\text{in}}(z)| \leq 1; \qquad |f(z)| \leq |f_{\text{out}}(z)|$$

  and on the unit circle ($|z| = 1$):

  $$|f_{\text{in}}(e^{i\theta})| = 1; \qquad |f(e^{i\theta})| = |f_{\text{out}}(e^{i\theta})| = [P(\theta)]^{1/2}$$

- Example of outer functions: phase states

$$\chi(z;\zeta) = \frac{(1-|\zeta|^2)^{1/2}}{1-\zeta z}; \qquad |\zeta| < 1$$

- Examples of inner functions: most general bilinear functions with $|f_{\text{in}}(e^{i\theta})| = 1$:

$$f(z) = \frac{\zeta - z}{1-\zeta^* z}; \qquad |\zeta| < 1$$

zero at $z = \zeta$
'building blocks' of more general inner functions.

- product of a finite number of such functions:

$$B(z;\{\zeta_N\}) = z^{p_0} \prod_{N=1}^{K} \left(\frac{\zeta_N - z}{1-\zeta_N^* z}\right)^{p_N}$$

ζ_N: distinct numbers in the unit disk.

Zeros of $f(z)$: Blaschke products

- zeros ζ_N of a bounded analytic function in the unit disc satisfy:

  $$\prod_{N=1}^{\infty} |\zeta_N|^{p_N} < \infty \leftrightarrow \sum_{N=1}^{\infty} p_N(1 - |\zeta_N|) < \infty \quad (1)$$

  p_N multiplicity of zero ζ_N.
  zeros move quickly towards the unit circle

  sequence ζ_N in D which violates Eq.(1) $\rightarrow$ corresponding phase states $\{\chi(z;\zeta_N)\}$ form an overcomplete set.

- sequence ζ_N in D which obeys Eq.(1) inner function with those zeros: Blaschke product

  $$B(z) = z^{p_0} \prod_{N=1}^{\infty} \left(\frac{\zeta_N^*}{|\zeta_N|} \frac{\zeta_N - z}{1 - \zeta_N^* z} \right)^{p_N}$$

  not unique answer: $B(z)\exp[h(z)]$

- general function $f_{\mathcal{Z}}(z)$ can be written as

  $f_{\mathcal{Z}}(z) = f_{\text{in}}(z) f_{\text{out}}(z) = B(z)\exp[h(z)] f_{\text{out}}(z)$

Discussion

- Analytic representation in the complex plane
  growth/density of zeros

- Analytic representation in the unit disc
  Bergmann space
  Hardy space

- finite Hilbert spaces (eg spin)
  complex functions $f(m)$ $m \in Z_d$
  stronger results when $m \in GF(p^\ell)$