Galois quantum systems

A. Vourdas
University of Bradford

A. Quantum systems with positions and momenta in $\mathbb{Z}(d)$

B. Quantum systems with positions and momenta in $GF(p^\ell)$

C. Analytic representations for finite systems

D. Other topics

Rep. Prog. Phys. 67 (2004) 267
J. Phys. A40, (2007) R285

Quantum systems with positions and momenta in $\mathbb{Z}(d)$

A1. Finite Fourier transform, position and momentum states, uncertainty relations

A2. displacements in the $\mathbb{Z}(d) \times \mathbb{Z}(d)$ phase space

A3. displaced parity operators

A4. symplectic transformations: isotropy of the $\mathbb{Z}(d) \times \mathbb{Z}(d)$ phase space

A5. marginal properties and Radon transforms

A6. Wigner and Weyl functions

A7. Discussion

Finite Fourier transform, position and momentum states, uncertainty relations

- d-dimensional Hilbert space $\mathcal{H}$:
  orthonormal basis: position states $|X;m\rangle$,
  $m \in \mathbb{Z}(d)$

- Fourier transform:

$$\begin{aligned} F &= d^{-1/2}\sum_{m,n}\omega(mn)|X;m\rangle\langle X;n|; \;\; \omega(\alpha) = \exp\left[i\frac{2\pi\alpha}{d}\right] \\ F^4 &= \mathbf{1} \end{aligned}$$

- eigenvalues of F: $1, i, -1, -i$

$$F = \varpi_0 + i\varpi_1 - \varpi_2 - i\varpi_3$$

  ϖ_λ projectors to eigenspaces

$$\varpi_\lambda = \sum_{\mu=0}^{3} i^{-\lambda\mu}F^\mu; \qquad F\varpi_\lambda = i^\lambda \varpi_\lambda$$

$$\varpi_\lambda\varpi_\nu = \varpi_\lambda\delta(\lambda,\nu); \qquad \varpi_0 + \varpi_1 + \varpi_2 + \varpi_3 = \mathbf{1}$$

- momentum states $|P;m\rangle$:

$$|P;m\rangle = F|X;m\rangle = d^{-1/2}\sum_n \omega(mn)|X;n\rangle$$

- acting with F:

  $|X;m\rangle \to |P;m\rangle \to |X;-m\rangle \to |P;-m\rangle \to |X;m\rangle$

- Position and momentum operators:

$$\begin{aligned}\widehat{x} &= \sum n|X;n\rangle\langle X;n|; \quad \widehat{p} = \sum n|P;n\rangle\langle P;n| \\ F\widehat{x}F^\dagger &= \widehat{p}; \qquad F\widehat{p}F^\dagger = -\widehat{x}\end{aligned}$$

  $[\widehat{x},\widehat{p}]$ discussed later

- arbitrary state $|s\rangle$:

$$|s\rangle = \sum \lambda_m |X;m\rangle = \sum \mu_n |P;n\rangle$$

$$\lambda_m = d^{-1/2}\sum \mu_n \omega(mn)$$

  λ_m wavefunction in the position repr.
  μ_n wavefunction in the momentum repr.

- **entropic uncertainty relations:**
  narrow $\lambda_m \to$ wide μ_n

$$\begin{aligned}\sigma_n &= \langle X;n|\rho|X;n\rangle; \quad \tau_n = \langle P;n|\rho|P;n\rangle \\ S_X &= -\sum \sigma_n \ln \sigma_n; \quad S_P = -\sum \tau_n \ln \tau_n \\ S_X &+ S_P \geq \ln D\end{aligned}$$

Displacements in the $\mathbb{Z}(d) \times \mathbb{Z}(d)$ phase space

- in harmonic oscillator plane ($R \times R$)
  continuous displacements with operators exp(iAx) and exp(iBp) where $A, B \in R$.

- In finite systems: phase space $\mathbb{Z}(d) \times \mathbb{Z}(d)$ toroidal lattice.
  discrete displacements with $Z(\alpha)$ and $X(\beta)$ where α, $\beta \in \mathbb{Z}(d)$.

$$
\begin{aligned}
X(\beta) &= \omega(-\beta\widehat{p}) = \exp\left[-i\frac{2\pi\beta}{d}\widehat{p}\right] \\
Z(\alpha) &= \omega(\alpha\widehat{x}) = \exp\left[i\frac{2\pi\alpha}{d}\widehat{x}\right]
\end{aligned}
$$

- $$
  \begin{aligned}
  Z(\alpha)|P; m\rangle &= |P; m+\alpha\rangle \\
  Z(\alpha)|X; m\rangle &= \omega(\alpha m)|X; m\rangle
  \end{aligned}
  $$

  and

  $$
  \begin{aligned}
  X(\beta)|P; m\rangle &= \omega(-m\beta)|P; m\rangle \\
  X(\beta)|X; m\rangle &= |X; m+\beta\rangle
  \end{aligned}
  $$

- $$X(d) = Z(d) = 1; \quad X(\beta)Z(\alpha) = Z(\alpha)X(\beta)\omega(-\alpha\beta)$$

- **below: odd d (Bose sector)**
  see references for even d (Fermi sector)

- displacement operators

$$D(\alpha,\beta) \equiv Z(\alpha)X(\beta)\omega(-2^{-1}\alpha\beta)$$
$$[D(\alpha,\beta)]^{\dagger} = D(-\alpha,-\beta)$$

  Heisenberg-Weyl group:
  $D(\alpha,\beta)\omega(\gamma)$ where $\alpha,\beta,\gamma \in \mathbb{Z}(d)$.

$$\begin{aligned} D(\alpha_1,\beta_1)D(\alpha_2,\beta_2) &= D(\alpha_1+\alpha_2,\beta_1+\beta_2) \\ &\times \omega[2^{-1}(\alpha_1\beta_2-\alpha_2\beta_1)] \end{aligned}$$

  2^{-1} exists for odd d

- $$D(\alpha,\beta)\widehat{x}[D(\alpha,\beta)]^{\dagger} = \widehat{x} - \beta\mathbf{1}$$
$$D(\alpha,\beta)\widehat{p}[D(\alpha,\beta)]^{\dagger} = \widehat{p} - \alpha\mathbf{1}$$

- $$\begin{aligned} D(\alpha,\beta)|X;m\rangle &= \omega[2^{-1}\alpha\beta+\alpha m]|X;m+\beta\rangle; \\ D(\alpha,\beta)|P;m\rangle &= \omega[-2^{-1}\alpha\beta-\beta m]|P;m+\alpha\rangle \end{aligned}$$

- Heisenberg-Weyl group discrete
  no Lie algebra; $[\widehat{x},\widehat{p}]$ no important role

$$\begin{aligned} [\widehat{x},\widehat{p}] &= (n-m)d\Delta_1(n-m)|X;n\rangle\langle X;m| \\ \Delta_1(x) &= \sum_{n=0}^{d-1} n\omega(nx) \end{aligned}$$

Displaced parity operators

- The parity operator $P(0,0)$ around the origin

$$P(0,0) = F^2; \qquad [P(0,0)]^2 = \mathbf{1}$$
$$P(0,0)|X;m\rangle = |X;-m\rangle; \qquad P(0,0)|P;m\rangle = |P;-m\rangle$$

eigenvalues $1, -1$

$$P(0,0) = (\varpi_0 + \varpi_2) - (\varpi_1 + \varpi_3)$$

- The displaced parity operators

$$\begin{aligned} P(\alpha,\beta) &= D(\alpha,\beta)P(0,0)[D(\alpha,\beta)]^\dagger \\ &= D(2\alpha,2\beta)P(0,0) = P(0,0)[D(2\alpha,2\beta)]^\dagger \end{aligned}$$

- $$[P(\alpha,\beta)]^2 = \mathbf{1}$$
$$P(\alpha,\beta) = \sum_{\gamma,\delta} D(\gamma,\delta)\omega(\alpha\delta - \beta\gamma)$$

Symplectic transformations: isotropy of the $\mathbb{Z}(d) \times \mathbb{Z}(d)$ phase space

- In harmonic oscillator (Bogoliubov trans.):

  $x' = \kappa x + \lambda p; \qquad p' = \mu x + \nu p; \qquad \kappa\nu - \lambda\mu = 1$

  preserve the commutation relations

  $$[x', p'] = [x, p] = i\mathbf{1}$$

  $Sp(2, R)$ group. 3 generators x^2, p^2, xp.
  In Quantum Optics 'squeezing'

- In finite systems:
  displacements along new axes:

  $$Z_{(q,r,s)}(\alpha) = S(q,r,s)\ Z(\alpha)\ S^{\dagger}(q,r,s) = D(t\alpha, s\alpha)$$
  $$X_{(q,r,s)}(\beta) = S(q,r,s)\ X(\beta)\ S^{\dagger}(q,r,s) = D(r\beta, q\beta)$$
  $$q, r, s, t \in \mathbb{Z}(d)$$

  require they preserve

  $$X_{(q,r,s)}(\beta) Z_{(q,r,s)}(\alpha) = Z_{(q,r,s)}(\alpha) X_{(q,r,s)}(\beta) \omega(-\alpha\beta)$$
  $$X_{(q,r,s)}(d) = Z_{(q,r,s)}(d) = \mathbf{1}$$

  leads to constraint

  $$qt - rs = 1$$

  3 independent parameters

- For a given triplet (q, r, s) can we find integer t such that $t = q^{-1}(rs + 1)$ $(mod\ d)$?
  Yes, if $d = p$ where p is prime.
  $\mathbb{Z}(p)$ Galois field (all non zero elements have inverse).
  Phase space $\mathbb{Z}(p) \times \mathbb{Z}(p)$ **finite geometry**.

  more generally when $d = p^n$
  $\alpha, \beta, \gamma, \delta \in GF(p^\ell)$
  'Galois quantum systems'
  discussed later

- $Sp(2, \mathbb{Z}(p))$ group
  construct explicitly symplectic operator S

$$\begin{aligned} S(q,r,s) &= S(1,0,\xi_1)S(1,\xi_2,0)S(\xi_3,0,0) \\ \xi_1 &= qs(1+rs)^{-1} \\ \xi_2 &= rq^{-1}(1+rs) \\ \xi_3 &= q(1+rs)^{-1} \end{aligned}$$

where

$$\begin{aligned} S(\xi_3,0,0) &= \sum_m |X;\xi_3 m\rangle\langle X;m| \\ S(1,\xi_2,0) &= \sum_m \omega(2^{-1}\xi_2 m^2)|X;m\rangle\langle X;m| \\ S(1,0,\xi_1) &= \sum_m \omega(-2^{-1}\xi_1 m^2)|P;m\rangle\langle P;m| \end{aligned}$$

- $S(\xi_3, 0, 0)$ dilation/contraction (squeezing) transformations

$$\begin{aligned} S(\xi_3, 0, 0)|\mathcal{X}; m\rangle &= |\mathcal{X}; \xi_3 m\rangle \\ S(\xi_3, 0, 0)|\mathcal{P}; m\rangle &= |\mathcal{P}; \xi_3^{-1} m\rangle \end{aligned}$$

take $d = 5$ and $\xi_3 = 3$ (then $3^{-1} = 2$)

$$\begin{aligned} S|X;0\rangle &= |X;0\rangle; \quad S|P;0\rangle = |P;0\rangle \\ S|X;1\rangle &= |X;3\rangle; \quad S|P;1\rangle = |P;2\rangle \\ S|X;2\rangle &= |X;1\rangle; \quad S|P;2\rangle = |P;4\rangle \\ S|X;3\rangle &= |X;4\rangle; \quad S|P;3\rangle = |P;1\rangle \\ S|X;4\rangle &= |X;2\rangle; \quad S|P;4\rangle = |P;3\rangle \end{aligned}$$

Reordering of points

Marginal properties and Radon transforms

- for **odd** d

$$\frac{1}{d}\sum_{\alpha\in\mathbb{Z}(d)} D(\alpha,\beta) = |X; 2^{-1}\beta\rangle\langle X; -2^{-1}\beta|$$

$$\frac{1}{d}\sum_{\beta\in\mathbb{Z}(d)} D(\alpha,\beta) = |P; 2^{-1}\alpha\rangle\langle P; -2^{-1}\alpha|$$

$$\frac{1}{d}\sum_{\alpha,\beta\in\mathbb{Z}(d)} D(\alpha,\beta) = P(0,0)$$

and also

$$\frac{1}{d}\sum_{\alpha\in\mathbb{Z}(d)} P(\alpha,\beta) = |X; \beta\rangle\langle X; \beta|$$

$$\frac{1}{d}\sum_{\beta\in\mathbb{Z}(d)} P(\alpha,\beta) = |P; \alpha\rangle\langle P; \alpha|$$

$$\frac{1}{d}\sum_{\alpha,\beta\in\mathbb{Z}(d)} P(\alpha,\beta) = \mathbf{1}$$

- in the Galois case $d = p$:
  isotropy of the $\mathbb{Z}(p) \times \mathbb{Z}(p)$ phase space
  'rotate' with $S(q, r, s)$:

$$\frac{1}{p} \sum_{\epsilon,\zeta} D(\epsilon, \zeta)\ \delta(-s\epsilon + t\zeta, \beta) =$$
$$|X(q, r, s); 2^{-1}\beta\rangle\langle X(q, r, s); -2^{-1}\beta|$$

  and also

$$\frac{1}{p} \sum_{\epsilon,\zeta} D(\epsilon, \zeta)\ \delta(q\epsilon - r\zeta, \alpha) =$$
$$|P(q, r, s); 2^{-1}\alpha\rangle\langle P(q, r, s); -2^{-1}\alpha|$$

  Radon transform: summation along the lines $-s\epsilon + t\zeta = \beta$ and $q\epsilon - r\zeta = \alpha$

  '$X(q, r, s)$-states' and '$P(q, r, s)$-states':

$$\begin{aligned} |X(q, r, s); \gamma\rangle &= S(q, r, s)|X; \gamma\rangle \\ |P(q, r, s); \gamma\rangle &= S(q, r, s)|P; \gamma\rangle \end{aligned}$$

- $$\begin{aligned}\frac{1}{p}\sum_{\epsilon,\zeta} P(\epsilon,\zeta)\;\delta(-s\epsilon+t\zeta,\beta) &= |X(q,r,s);\beta\rangle\langle X(q,r,s);\beta| \\ \frac{1}{p}\sum_{\epsilon,\zeta} P(\epsilon,\zeta)\;\delta(q\epsilon-r\zeta,\alpha) &= |P(q,r,s);\alpha\rangle\langle P(q,r,s);\alpha|\end{aligned}$$

- inverse Radon transform

$$\begin{aligned}D(r\beta,q\beta) &= \sum_{\alpha\in\mathbb{Z}(p)} |P(q,r,s);\alpha\rangle\langle P(q,r,s);\alpha|\;\omega(-\alpha\beta) \\ D(t\alpha,s\alpha) &= \sum_{\beta\in\mathbb{Z}(p)} |X(q,r,s);\beta\rangle\langle X(q,r,s);\beta|\;\omega(\alpha\beta)\end{aligned}$$

- generalised resolution of the identity

$$\frac{1}{p}\sum_{\alpha,\beta} D(\alpha,\beta)\;\frac{\Theta}{\mathrm{tr}\Theta}\;[D(\alpha,\beta)]^\dagger = \mathbf{1}$$

special case $\Theta = |s\rangle\langle s|$

$$\frac{1}{p}\sum_{\alpha,\beta} |\alpha,\beta;s\rangle\langle\alpha,\beta;s| = \mathbf{1}; \qquad |\alpha,\beta;s\rangle \equiv D(\alpha,\beta)|s\rangle$$

The p^2 states $|\alpha,\beta;s\rangle$ overcomplete set of states in the p-dim Hilbert space

Wigner and Weyl functions

- use notation:

  $$\Theta_X(\alpha,\beta) = \langle X;\alpha|\Theta|X;\beta\rangle; \quad \Theta_P(\alpha,\beta) = \langle P;\alpha|\Theta|P;\beta\rangle$$

- **Wigner functions** Θ operator:

  $$\begin{aligned} W(\Theta;\alpha,\beta) &= \mathrm{Tr}[\Theta P(\alpha,\beta)] \\ &= \sum_{\ell} \omega(2\alpha\beta - 2\alpha\ell)\Theta_X(\ell, 2\beta-\ell) \end{aligned}$$

  Θ Hermitian $\rightarrow$ Wigner real **pseudoprobability** distr. (for density matrices)

- 'marginal properties'(d:odd):

  $$\begin{aligned} \frac{1}{d}\sum_{\beta=0}^{d-1} W(\Theta;\alpha,\beta) &= \Theta_P(\alpha,\alpha) \\ \frac{1}{d}\sum_{\alpha=0}^{d-1} W(\Theta;\alpha,\beta) &= \Theta_X(\beta,\beta) \\ \frac{1}{d}\sum_{\alpha,\beta} W(\Theta;\alpha,\beta) &= \mathrm{Tr}\Theta \end{aligned}$$

  for prime d: marginal properties with respect to all axes (isotropy of $\mathbb{Z}(p)\times\mathbb{Z}(p)$ phase space)

- **Weyl function: generalized correlation function**
  α, β: position and momentum **increments**
  operator Θ,

$$\begin{aligned}\tilde{W}(\Theta;\alpha,\beta) &\equiv \text{Tr}[\Theta D(\alpha,\beta)] \\ &= \sum_{\ell} \omega(2^{-1}\alpha\beta+\alpha\ell)\Theta_X(\ell,\beta+\ell)\end{aligned}$$

- Weyl: Fourier transform of Wigner

$$\tilde{W}(\Theta;\alpha,\beta)=\frac{1}{d}\sum_{\gamma,\delta} W(\Theta;\gamma,\delta)\omega(\alpha\delta-\beta\gamma)$$

- 'marginal properties' (d: odd)

$$\begin{aligned}\frac{1}{d}\sum_{\beta=0}^{d-1}\tilde{W}(\Theta;\alpha,\beta) &= \Theta_P(-2^{-1}\alpha,2^{-1}\alpha) \\ \frac{1}{d}\sum_{\alpha=0}^{d-1}\tilde{W}(\Theta;\alpha,\beta) &= \Theta_X(-2^{-1}\beta,2^{-1}\beta) \\ \frac{1}{d}\sum_{\alpha,\beta}\tilde{W}(\Theta;\alpha,\beta) &= W(\Theta;0,0)\end{aligned}$$

  for prime d: marginal properties with respect to all axes (isotropy of $\mathbb{Z}(p)\times\mathbb{Z}(p)$ phase space)

- quantum tomography

$$\tilde{W}(\Theta; t\alpha, s\alpha) = \sum_{\beta} \langle X(q,r,s); \beta|\rho|X(q,r,s); \beta\rangle\, \omega(\alpha\beta)$$
$$\tilde{W}(\Theta; r\beta, q\beta) = \sum_{\alpha} \langle P(q,r,s); \alpha|\rho|P(q,r,s); \alpha\rangle\, \omega(-\alpha\beta)$$

- arbitrary operator Θ:

$$\Theta = \frac{1}{d}\sum_{\alpha,\beta} W(\Theta; \alpha, \beta) P(\alpha, \beta)$$
$$= \frac{1}{d}\sum_{\alpha,\beta} \tilde{W}(\Theta; -\alpha, -\beta) D(\alpha, \beta)$$

$D(\alpha, \beta)$ as generators of $SU(d)$ trans (later)

Discussion:

- phase-space formalism in the context of finite systems.

- $\mathbb{Z}(d) \times \mathbb{Z}(d)$ phase space. Displacements.

- Phase space finite geometry for 'Galois quantum systems' ($d = p$).

  Symplectic transformations. $Sp(2, \mathbb{Z}(p))$ group.

- Wigner and Weyl functions; quantum tomography

Quantum systems with positions and momenta in $GF(p^\ell)$

B1. Galois fields

B2. Galois quantum systems

B3. Displacements in $GF(p^\ell) \times GF(p^\ell)$ phase space

B4. Symplectic $Sp(2, GF(p^\ell))$ transformations

B5. Quantum systems with positions and momenta in a subfield $GF(p^d)$ of $GF(p^\ell)$

B6. Frobenius transformations: implications for physics

B7. Discussion

Galois fields

- $\mathbb{Z}(p)$: field
  Field extension: elements of $GF(p^\ell)$

  $$\alpha = \alpha_0 + \alpha_1\epsilon + ... + \alpha_{\ell-1}\epsilon^{\ell-1}; \quad \alpha_i \in \mathbb{Z}(p)$$

  defined modulo **irreducible** polynomial of degree ℓ:

  $$P(\epsilon) = c_0 + c_1\epsilon + ... + c_{\ell-1}\epsilon^{\ell-1} + \epsilon^\ell; \quad c_i \in \mathbb{Z}(p)$$

  different $P(\epsilon)$, isomorphic results
  addition, multiplication

- Frobenius automorphism:

  $$\sigma: \alpha \to \alpha^p; \qquad \sigma^\ell = \mathbf{1}$$
  $$\alpha \to \alpha^p \to \alpha^{p^2} \to ... \to \alpha^{p^{\ell-1}} \to \alpha^{p^\ell} = \alpha$$

  Galois conjugates: α, α^p,...$\alpha^{p^{\ell-1}}$

- Trace: sum of all conjugates

  $$\mathrm{Tr}\alpha = \alpha + \alpha^p + ... + \alpha^{p^{\ell-1}}; \quad \mathrm{Tr}\alpha \in \mathbb{Z}(p)$$

  trace of $\alpha\beta$ can be written as

  $$\mathrm{Tr}(\alpha\beta) = \sum g_{ij}\alpha_i\beta_j; \quad g_{ij} = \mathrm{Tr}[\epsilon^{i+j}]$$

  g_{ij}: depends on irreducible polynomial
  matrix g has inverse

- dual basis E_i such that $\mathrm{Tr}(\epsilon^\kappa E_\lambda) = \delta_{\kappa\lambda}$. Then

$$\begin{aligned} \alpha &= \sum_{\lambda=0}^{\ell-1} \alpha_\lambda \epsilon^\lambda = \sum_{\lambda=0}^{\ell-1} \bar{\alpha}_\lambda E_\lambda \\ \alpha_\lambda &= \mathrm{Tr}[\alpha E_\lambda]; \qquad \bar{\alpha}_\lambda = \mathrm{Tr}[\alpha \epsilon^\lambda] = \sum_\kappa g_{\lambda\kappa} \alpha_\kappa \end{aligned}$$

  trace of $\alpha\beta$ can be written as

$$\mathrm{Tr}(\alpha\beta) = \sum \bar{\alpha}_i \beta_i = \sum \alpha_i \bar{\beta}_i$$

- exponential of α (complex valued function):

$$\chi(\alpha) = \omega(\mathrm{Tr}\alpha); \quad \omega = \exp(i2\pi/p)$$

  additive characters: $\chi(\alpha)\chi(\beta) = \chi(\alpha+\beta)$
  later in Fourier transforms:

$$\chi(\alpha\beta) = \omega\left(\sum g_{ij}\alpha_i\beta_j\right) = \omega\left(\sum \bar{\alpha}_i\beta_i\right) = \omega\left(\sum \alpha_i\bar{\beta}_i\right)$$

  where

$$\sum_{\alpha \in GF(p^\ell)} \chi(\alpha\beta) = \delta(\beta, 0)$$

- example: $GF(9)$(where $p = 3$, $\ell = 2$)
  choose $P(\epsilon) = \epsilon^2 + \epsilon + 2$

$$g = \begin{pmatrix} -1 & -1 \\ -1 & 0 \end{pmatrix}$$

  off-diagonal elements: coupling between subsystems later

- **ring** $[\mathbb{Z}(p)]^\ell \equiv \mathbb{Z}(p) \times ... \times \mathbb{Z}(p)$
  Addition and multiplication:

  $(\alpha_\lambda) + (\beta_\lambda) = (\alpha_\lambda + \beta_\lambda); \qquad (\alpha_\lambda)(\beta_\lambda) = (\alpha_\lambda \beta_\lambda)$

  $(0, ..., 0)$ zero; and $(1, ..., 1)$ unity

- additive characters

$$\psi[(\alpha_\lambda)] = \omega\left(\sum_\lambda \alpha_\lambda\right)$$

  and

$$\frac{1}{p^\ell} \sum_{(\alpha_\lambda)} \psi[(\alpha_\lambda)(\beta_\lambda)] = \delta[(\beta_\lambda), (0)]$$

- **compare** harmonic analysis on $GF(p^\ell)$ with harmonic analysis on $[\mathbb{Z}_p]^\ell$
  at this stage compare

$$\psi[(\alpha_\lambda \beta_\lambda)] = \omega\left(\sum_\lambda \alpha_\lambda \beta_\lambda\right)$$

  with the

$$\chi(\alpha\beta) = \omega\left(\sum_{\lambda,\mu} g_{\lambda\mu} \alpha_\lambda \beta_\mu\right) = \omega\left(\sum_\lambda \overline{\alpha}_\lambda \beta_\lambda\right) = \omega\left(\sum_\lambda \alpha_\lambda \overline{\beta}_\lambda\right)$$

  g_{ij} related to Galois theory

Galois quantum systems

- p^ℓ-dimensional
  position (and momentum) in $GF(p^\ell)$
  motivation:

  1. symplectic transformations
  2. mutually unbiased bases
  3. Galois fields important in classical coding; transfer these techniques to quantum coding
  4. transfer field extension concept to Hilbert spaces; blend harmonic analysis with field extension

- tensor product of ℓ spaces:
  $H = \mathcal{H} \otimes ... \otimes \mathcal{H}$
  $\mathcal{H}$ is p-dimensional
  H is p^ℓ-dimensional
  e.g., ℓ **coupled** spins $j = (p-1)/2$

  in this space:

  - Galois systems with position/momentum in $GF(p^\ell)$
    position states in H

    $$|X; m\rangle \equiv |\mathcal{X}; m_0\rangle \otimes ... \otimes |\mathcal{X}; m_{\ell-1}\rangle$$
    $$m = \sum_i m_i \epsilon^i \ \in GF(p^\ell); \qquad m_i \in \mathcal{Z}(p)$$

  - 'R-systems' with position/momentum in the ring $\mathbb{Z}(p) \times ... \times \mathbb{Z}(p)$.
    Position states

    $$|X; (m_\lambda)\rangle \equiv |\mathcal{X}; m_0\rangle \otimes ... \otimes |\mathcal{X}; m_{\ell-1}\rangle$$

- Fourier transform in Galois systems:

$$F = (p^\ell)^{-1/2} \sum_{m,n} \chi(mn) |X; m\rangle\langle X; n|$$

$$= (p^\ell)^{-1/2} \sum_{m_i,n_j} \omega \left[\sum_{i,j} g_{ij} m_i n_j \right]$$

$$\times \quad |\mathcal{X}; m_0\rangle\langle\mathcal{X}; n_0| \otimes ... \otimes |\mathcal{X}; m_{\ell-1}\rangle\langle\mathcal{X}; n_{\ell-1}|$$

g_{ij} related to Galois theory

- Fourier transform in R-systems:

$$\mathcal{F} \otimes ... \otimes \mathcal{F} = (p^\ell)^{-1/2} \sum_{m_i,n_i} \omega \left[\sum_{i,j} m_i n_i \right]$$

$$\times \quad |\mathcal{X}; m_0\rangle\langle\mathcal{X}; n_0| \otimes ... \otimes |\mathcal{X}; m_{\ell-1}\rangle\langle\mathcal{X}; n_{\ell-1}|$$

independent Fourier tr. on each subsystem
different from F

- Hamiltonian for Galois systems:

$$h = h(\hat{Q}, \hat{P}) = h(\hat{Q}, F\hat{Q}F^{\dagger})$$

  special coupling between component systems related to off-diagonal g_{ij}
  in contrast:
  Hamiltonian for 'R-systems':

$$h' = h'(\hat{\mathcal{Q}}_0, \hat{\mathcal{P}}_0; ...; \hat{\mathcal{Q}}_{\ell-1}, \hat{\mathcal{P}}_{\ell-1})$$

  arbitrary coupling between component systems
  h **very special case of** h'

- momentum states in Galois systems:

$$|P; m\rangle = F|X; m\rangle = |\mathcal{P}; \bar{m}_0\rangle \otimes ... \otimes |\mathcal{P}; \bar{m}_{\ell-1}\rangle$$
$$m = \sum_i m_i \epsilon^i = \sum_i \bar{m}_i E_i$$

  momentum states in R-systems:

$$|P; m\rangle = \mathcal{F} \otimes ... \otimes \mathcal{F}|X; m\rangle = |\mathcal{P}; m_0\rangle \otimes ... \otimes |\mathcal{P}; m_{\ell-1}\rangle$$

Displacements in $GF(p^\ell) \times GF(p^\ell)$ phase space

- displacement operators: as before **with trace**

$$Z(\alpha) \equiv \sum_n \chi(\alpha n)|X;n\rangle\langle X;n|$$

- $GF(p^\ell) \times GF(p^\ell)$ phase space:
general displacements

$$D(\alpha,\beta) = Z(\alpha)X(\beta)\chi(-2^{-1}\alpha\beta)$$

relationship between displacement operator and displacement operators in component systems

$$D(\alpha,\beta) = \mathcal{D}(\bar{\alpha}_0,\beta_0) \otimes ... \otimes \mathcal{D}(\bar{\alpha}_{\ell-1},\beta_{\ell-1})$$

- displaced parity operator

$$P(\alpha,\beta) = \mathcal{P}(\bar{\alpha}_0,\beta_0) \otimes ... \otimes \mathcal{P}(\bar{\alpha}_{\ell-1},\beta_{\ell-1})$$

- Fourier transform between $P(\alpha,\beta)$ and $D(\gamma,\delta)$ **with trace**:

$$P(\alpha,\beta) = \sum_{\gamma,\delta} D(\gamma,\delta)\chi(\alpha\delta - \beta\gamma)$$

Symplectic $Sp(2, GF(p^\ell))$ transformations

previous formulas **with trace**
uses Galois multiplication
First discuss 3 subgroups:

- first subgroup:

$$S(\xi,0,0) = \sum_{m\in GF(p^\ell)} |X;\xi m\rangle\langle X;m| = \sum_{m\in GF(p^\ell)} |P;\xi^{-1}m\rangle\langle P;m|$$

  where

$$S(\xi_1,0,0)\, S(\xi_2,0,0) = S(\xi_1\xi_2,0,0)$$
$$[S(\xi,0,0)]^{p^\ell} = S(\xi,0,0)$$

- second subgroup:

$$S(1,\xi,0) = \sum_{m\in GF(p^\ell)} \chi(2^{-1}\xi m^2)|X;m\rangle\langle X;m|$$

  where

$$S(1,\xi_1,0)\, S(1,\xi_2,0) = S(1,\xi_1+\xi_2,0)$$
$$[S(1,\xi,0)]^p = \mathbf{1}$$

- third subgroup:

$$S(1,0,\xi) = \sum_{m\in GF(p^\ell)} \chi(-2^{-1}\xi m^2)|P;m\rangle\langle P;m|$$

where

$$\begin{aligned} &S(1,0,\xi_1)\, S(1,0,\xi_2) = S(1,0,\xi_1+\xi_2) \\ &[S(1,0,\xi)]^p = \mathbf{1} \\ &S(1,0,\xi) = FS(1,\xi,0)F^\dagger \end{aligned}$$

- general symplectic operator

$$\begin{aligned} S(q,r,s) &= S(1,0,\xi_1)\, S(1,\xi_2,0)\, S(\xi_3,0,0) \\ \xi_1 &= qs(1+rs)^{-1} \\ \xi_2 &= rq^{-1}(1+rs) \\ \xi_3 &= q(1+rs)^{-1} \end{aligned}$$

-

$$\begin{aligned} S(q,r,s) &= p^{-\ell}\, G(A) \sum_{n,m\in GF(p^\ell)} \chi[(2q)^{-1}(s^{-1}+r)n^2 \\ &- s^{-1}nm + (2s)^{-1}qm^2]|X;n\rangle\langle X;m| \\ A &= -2^{-1}(1+rs)^{-1}qs \end{aligned}$$

$G(A)$ is the Gauss sum

$$G(A) = \sum_{r\in GF(p^\ell)} \chi(Ar^2)$$

- semidirect product of Heisenberg-Weyl $HW[GF(p^\ell)]$ group by the symplectic $Sp[2, GF(p^\ell)]$ group: larger group with both displacements and symplectic transformations

- Wigner and Weyl functions
  marginal properties with respect to all axes: isotropy of $GF(p^\ell) \times GF(p^\ell)$ phase space

Quantum systems with positions and momenta in a subfield $GF(p^d)$ of $GF(p^\ell)$

- if $d|\ell$ then $GF(p^d)$ subfield of $GF(p^\ell)$

  $$H_d = \text{span}\{|X; m\rangle : \ m \in GF(p^d)\} \subset H$$

  important to understand the relationship of the full Galois system in H with the Galois subsystem in H_d

- example with $d = 1$

  $$H_1 = \text{span}\{|X; m\rangle : \ m \in \mathbb{Z}(p)\} \subset H$$

- relationship between
  Fourier, displacements and symplectic in H
  with
  Fourier, displacements and symplectic in H_d

Frobenius transformations

- positions have the Frobenius property

  $$\alpha \ \rightarrow \ \alpha^p \ \rightarrow \ \alpha^{p^2} \ \rightarrow ... \rightarrow \ \alpha^{p^{\ell-1}} \ \rightarrow \ \alpha^{p^\ell} = \alpha$$

  implications for physics

- Frobenius transformations:

  $$\mathcal{G} \equiv \sum_m |X; m^p\rangle\langle X; m| = \sum_m |P; m^p\rangle\langle P; m|$$

  $$\mathcal{G}\mathcal{G}^\dagger = \mathbf{1}; \quad \mathcal{G}^\ell = \mathbf{1}; \quad [\mathcal{G}, F] = 0$$

- Galois Group:

  $$\mathrm{Gal}[H/H_1] = \{\mathbf{1}, \mathcal{G}, ..., \mathcal{G}^{\ell-1}\}$$

  cyclic group of order ℓ
  leaves fixed all states in the subspace H_1

- Galois group:

  $$\mathrm{Gal}[H/H_d] = \{\mathbf{1}, \mathcal{G}^d, ..., \mathcal{G}^{\ell-d}\}$$

  subgroup of $\mathrm{Gal}[H/H_1]$
  cyclic group of order d
  leaves fixed all states in the p^d-dim subsp. H_d
  if Π_d the projector to H_d then

  $$\mathcal{G}^d \Pi_d = \Pi_d$$

  for $d = \ell$ this becomes $\mathcal{G}^\ell = \mathbf{1}$

- $$\begin{aligned} \mathcal{G}^{\lambda}|X;m\rangle &= |X;m^{p^{\lambda}}\rangle \\ \mathcal{G}^{\lambda}|P;m\rangle &= |P;m^{p^{\lambda}}\rangle \\ \mathcal{G}^{\lambda}D(\alpha,\beta)(\mathcal{G}^{\dagger})^{\lambda} &= D\left(\alpha^{p^{\lambda}},\beta^{p^{\lambda}}\right) \end{aligned}$$

  for $\alpha,\beta \in \mathcal{Z}_p$ the $\mathcal{G}$ commutes with $D(\alpha,\beta)$.

- if $\mathcal{G}$ commutes with the Hamiltonian h:
  $\ell - 1$ constants of motion

  $$\mathrm{tr}[\rho(t)\varpi(\lambda)] = \mathrm{tr}[\rho(0)\varpi(\lambda)]$$

- example: $GF(9)$(where $p=3$, $\ell=2$)
  choose $P(\epsilon)=\epsilon^2+\epsilon+2$

$$\mathcal{G}=\varpi(0)-\varpi(1)$$

  If hamiltonian commutes with $\mathcal{G}$

$$\begin{aligned}
\mathrm{tr}[\rho\varpi(1)] &= \frac{1}{2}[\rho_X(\epsilon,\epsilon)+\rho_X(1+\epsilon,1+\epsilon) \\
&+ \rho_X(2+\epsilon,2+\epsilon)+\rho_X(1+2\epsilon,1+2\epsilon) \\
&+ \rho_X(2\epsilon,2\epsilon)+\rho_X(2+2\epsilon,2+2\epsilon) \\
&- \rho_X(2+2\epsilon,\epsilon)-\rho_X(\epsilon,2+2\epsilon) \\
&- \rho_X(2\epsilon,1+\epsilon)-\rho_X(1+\epsilon,2\epsilon) \\
&- \rho_X(1+2\epsilon,2+\epsilon)-\rho_X(2+\epsilon,1+2\epsilon)]
\end{aligned}$$

  constant in time
  Notation:

$$\rho_X(m,n)=\langle X;m|\rho|X;n\rangle$$

  $\mathrm{tr}[\rho\varpi(0)]$ also constant in time (not independent)

- **Frobenius symmetry: discrete cyclic symmetry with no analogue in harmonic oscillator**
  in this sense:
  $GF(p^\ell)\times GF(p^\ell)$ phase space more symmetry than the continuum $R\times R$

Discussion

- Galois quantum system:
  position and momentum in $GF(p^\ell)$

  Hilbert space $H = \mathcal{H} \otimes ... \otimes \mathcal{H}$
  subsystems coupled through the matrix g_{ij} which is related to Galois theory

  Fourier transform contains g_{ij}

  Hamiltonian

  $$h = h(\hat{Q}, \hat{P}) = h(\hat{Q}, F\hat{Q}F^\dagger)$$

  special coupling between component systems related to off-diagonal g_{ij}

- Displacements and symplectic transformations

- Frobenius transformations: discrete cyclic symmetry

Analytic representations for finite systems

C1a. Analytic representation of general systems with discrete symmetries, in the d-sheeted complex plane
motivation:
discrete cyclic symmetry $\leftrightarrow$ multivaluedness $\leftrightarrow$ Riemann surfaces

C1b. Analytic representation of Galois quantum systems with Frobenius symmetries, in the ℓ-sheeted complex plane
motivation:
Frobenius symmetry $\leftrightarrow$ multivaluedness $\leftrightarrow$ Riemann surface

C2. Theta function repr of d-dim systems on a torus (any d)
motivation:
theory of finite systems in the language of Theta functions

C3. Discussion

Analytic representation of general systems with discrete symmetries, in the d-sheeted complex plane

- harmonic oscillator (∞-dim Hilbert space H) number eigenstates:

$$|M\rangle_n = \frac{(a^\dagger)^M}{(M!)^{1/2}}|0\rangle_n$$

split Hilbert space as

$$\mathsf{H} = \bigoplus_{m=0}^{d-1} \mathbb{H}_m = \bigoplus_{N=0}^{\infty} \mathsf{H}_N$$
$$\mathbb{H}_m = \mathrm{span}\{|m\rangle_n, |d+m\rangle_n, |2d+m\rangle_n, ...\}$$
$$\mathsf{H}_N = \mathrm{span}\{|dN\rangle_n, |dN+1\rangle_n, ..., |dN+d-1\rangle_n\}$$

$\mathbb{P}(m)$ the projectors to $\mathbb{H}_m$ ($m \in \mathbb{Z}(d)$)

- Fourier transforms in H_N

$$\mathsf{F} = d^{-1/2} \sum_{N=0}^{\infty} \left[\sum_{m,k\in\mathbb{Z}(d)} \Omega_d(-mk)|m+dN\rangle_n \,{}_n\langle k+dN| \right]$$
$$\mathsf{F}^4 = \mathbf{1}$$

'dual number states'

$$|M\rangle_d = \mathsf{F}|M\rangle_n$$

- $\wp(m)$ the projectors:

$$\wp(m) = \mathsf{F}\mathbb{P}(m)\mathsf{F}^\dagger = \sum_{N=0}^{\infty} |m + dN\rangle_{d\ d}\langle m + dN|$$

- consider the unitary operator

$$\begin{aligned} \mathsf{G} &= \sum_{N=0}^{\infty} \left[\sum_{m \in \mathbb{Z}(d)} |dN + m + 1\rangle_{n\ n}\langle dN + m| \right] \\ &= \wp(0) + \Omega_d(1)\wp(1) + ... + \Omega_d(d-1)\wp(d-1) \end{aligned}$$

  then

$$\mathsf{G}^d = \mathsf{G}\mathsf{G}^\dagger = \mathbf{1}$$

- assume Hamiltonian h commutes with G:

$$[\mathsf{G}, \mathsf{h}] = 0$$

  system has discrete symmetry
  multivaluedness: if $|s\rangle$ is an eigenstate of h, then all states $\mathsf{G}^m|s\rangle$ are eigenstates with the same eigenvalue.

- density matrix $\rho(t)$
  probabilities $\mathrm{tr}[\rho(t)\wp(m)]$ are constant in time:

$$\mathrm{tr}[\rho(t)\wp(m)] = \mathrm{tr}[\rho(0)\wp(m)]$$

  $d - 1$ independent conserved quantities.

- $C^* = C - \{0\}$ punctured complex plane
  Riemann surface $C^*/\mathbb{Z}(d)$
  its covering surface is the d-sheeted complex plane with the cuts T_m

  $$T_m = \{z = r\Omega_d(m);\ \ r \geq 0\}\,; \qquad m \in \mathbb{Z}(d)$$

  and the sheets

  $$\Xi_m = \left\{ z = r\,\mathsf{exp}(i\phi);\ \ r \geq 0;\ \frac{2\pi m}{d} < \phi < \frac{2\pi(m+1)}{d} \right\}$$

  sheet number of z:

  $$\tau(z;d) = \mathsf{IP}\left(\frac{d\,\mathsf{arg}(z)}{2\pi}\right); \qquad \tau(z;d) \in \mathbb{Z}(d)$$

  IP:integer part

- state in H:

$$|s\rangle = \sum_{M=0}^{\infty} s(M)|M\rangle_n; \qquad \sum_{M=0}^{\infty} |s(M)|^2 = 1$$

To each M corresponds a pair (m, N):

$$M = Nd + m; \qquad N \in \mathbb{Z}; \qquad m \in \mathbb{Z}(d)$$

represent

$$s(M)|M\rangle_n \;\; \rightarrow \;\; s(M)\, z^{Nd}\, (N!)^{-1/2}$$

non-zero only in the m-sheet

In the m-sheet only the projection $\mathbb{P}_m|s\rangle$
in all the d sheets the full state

the full state

$$|s\rangle \;\; \rightarrow \;\; S_1(z) = \sum_{N=0}^{\infty} s[\tau(z;d) + Nd]\, z^{Nd}\, (N!)^{-1/2}$$

$S_1(z)$ analytic in the interior of all Ξ_m
discontinuities across the cuts T_m

- transformations G implemented as

$$\mathsf{G}^m|s\rangle \;\; \to S_1[\Omega_d(-m)z]; \qquad \Omega_d(\alpha) = \exp\left(\frac{i2\pi\alpha}{d}\right)$$

$\mathsf{G}^d = \mathbf{1}$ related to $S_1[\Omega_d(-d)z] = S_1[z]$

- scalar product

$$\begin{aligned}\langle s|r\rangle &= \int_C d\mu_d(z)\ \exp(-|z|^{2d})\ [S_1(z)]^*\ R_1(z) \\ &= \sum_{M=0}^{\infty} [s(M)]^* r(M) \\ d\mu_d(z) &= d^2|z|^{2(d-1)}\frac{dz_R dz_I}{\pi}\end{aligned}$$

- another analytic representation:
  ℓ-sheeted complex plane where $d|\ell$.

  sheet number modulo d:

$$\tau(z;\ell;\text{mod } d) = \tau(z;\ell) \pmod d$$
$$\tau(z;\ell;\text{mod } d) \in \mathbb{Z}_d$$

  ℓ/d sheets with same number $\tau(z;\ell;\text{mod } d)$.

  represent the state as before (with d/ℓ)
  first d sheets: the full state
  periodicity after that:

$$S_2[z\Omega_\ell(d)] = S_2(z)$$

  express this as

$$|s\rangle \;\; \rightarrow \;\; S_2(z) = \frac{d}{\ell}\sum_{N=0}^{\infty} s[\tau(z;\ell;\text{mod } d) + Nd]\; z^{N\ell}\,(N!)^{-1/2}$$

- transformations G implemented as:

$$\mathsf{G}^m|s\rangle \;\; \rightarrow S_2[z\Omega_\ell(-m)]$$

  $\mathsf{G}^d = \mathbf{1}$ related to $S_2[z\Omega_\ell(d)] = S_2(z)$.

- scalar product

$$\langle s|r\rangle = \int_C d\mu_\ell(z)\;\exp(-|z|^{2\ell})\;[S_1(z)]^*\,R_1(z)$$
$$d\mu_\ell(z) = \ell^2|z|^{2(\ell-1)}\frac{dz_R dz_I}{\pi}$$

Analytic representation of Galois quantum systems with Frobenius symmetries, in the ℓ-sheeted complex plane

- in $GF(p^\ell)$:

$$y^{p^\ell} - y = \prod_{\mathfrak{N}=1}^{\mathfrak{M}(\ell,p)} f_{\mathfrak{N}}(y)$$

irred. polyn. of order $d|\ell$ ($d = 1, ..., \ell$)
$\mathfrak{N} = (d, \kappa)$ label of the irred. polyn.
label elements in $GF(p^\ell)$ as $m(\mathfrak{N}, \nu)$ (or $m(d, \kappa, \nu)$)

- example: $GF(9)$, $p = 3$, $\ell = 2$
choose $P(\epsilon) = \epsilon^2 + \epsilon + 2$

$$y^9 - y = \prod_{i=1}^{6} f_i(y)$$
$$f_1(y) = f_{11}(y) = y$$
$$f_2(y) = f_{12}(y) = y - 1$$
$$f_3(y) = f_{13}(y) = y - 2$$
$$f_4(y) = f_{21}(y) = y^2 + 2y + 2 = (y - 1 - 2\epsilon)(y - 2 - \epsilon)$$
$$f_5(y) = f_{22}(y) = y^2 + y + 2 = (y - \epsilon)(y - 2 - 2\epsilon)$$
$$f_6(y) = f_{23}(y) = y^2 + 1 = (y - 1 - \epsilon)(y - 2\epsilon)$$

$\mathfrak{H}_{\mathfrak{N}}$ corresponding Hilbert spaces, e.g.,

$$\mathfrak{H}_4 = \text{span}\{|X; 1 + 2\epsilon\rangle, |X; 2 + \epsilon\rangle\}$$

- general state

$$|s\rangle = \sum_{\mathfrak{N}=1}^{\mathfrak{M}(\ell,p)} \sum_{\nu=1}^{d} s(\mathfrak{N},\nu)|X; m(\mathfrak{N},\nu)\rangle$$

consider projection to the d-dim space $\mathfrak{H}_{\mathfrak{N}}$:

$$\pi_{\mathfrak{N}}|s\rangle = \sum_{\nu=1}^{d} s(\mathfrak{N},\nu)|X; m(\mathfrak{N},\nu)\rangle$$

represent it in the ℓ-sheeted complex plane (where $d|\ell$) with the function

$$\mathfrak{S}_{\mathfrak{N}}(z) = \sum_{\nu=1}^{d} \frac{d}{\ell}\, s\left[\mathfrak{N}, \nu = \tau(z;\ell;\text{mod } d)\right]\; z^{\ell\mathfrak{N}}\,(\mathfrak{N}!)^{-1/2}$$

- Frobenius transformations $\mathcal{G}$

$$\mathcal{G}^m \pi_{\mathfrak{N}}|s\rangle \;\;\rightarrow\;\; \mathfrak{S}_{\mathfrak{N}}[z\Omega_\ell(-m)]$$

$\mathcal{G}^d \pi_{\mathfrak{N}} = \pi_{\mathfrak{N}}$ related to $\mathfrak{S}_{\mathfrak{N}}[z\Omega_\ell(d)] = \mathfrak{S}_{\mathfrak{N}}(z)$

- full state $|s\rangle$

$$\mathfrak{S}(z) = \sum_{\mathfrak{N}=1}^{\mathfrak{M}(\ell,p)} \frac{d}{\ell}\, s\left[\mathfrak{N}, \nu = \tau(z;\ell;\mathrm{mod}\ d)\right]\ z^{\ell\mathfrak{N}}\,(\mathfrak{N}!)^{-1/2}$$

- Frobenius transformations

$$\mathcal{G}^m|s\rangle \;\;\rightarrow\;\; \mathfrak{S}[z\Omega_\ell(-m)]$$

$\mathcal{G}^d\pi_\mathfrak{N} = \pi_\mathfrak{N}$ and more generally $\mathcal{G}^d\Pi_d = \Pi_d$ (where $\Pi_d = \sum \pi_\mathfrak{N}$) in periodic structure of formalism

relations $\mathcal{G}^d\Pi_d = \Pi_d$ in Galois subsystems embodied naturally in periodic structure of formalism (for all $d|\ell$).

Theta function repr of d-dim systems on a torus (any d)

- the state

$$|f\rangle = \sum_m f_m |\mathcal{X};\ m\rangle$$

is represented with

$$\begin{aligned} f_T(z) &= \pi^{-1/4} \sum_m f_m\, \Theta_3 \left[-\pi^2 m A^{-1} + z\pi A^{-1/2};\ \frac{i}{d}\right] \\ A &= \pi d \end{aligned}$$

analytic and quasi-periodic (**not periodic**)

$$\begin{aligned} f_T\left[z + A^{1/2}\right] &= f_T(z) \\ f_T\left[z + iA^{1/2}\right] &= f_T(z) \exp\left[A - 2iA^{1/2} z\right]. \end{aligned}$$

Consider it on a cell of area A:

$$S = \left[a, a + A^{1/2}\right)_R \times \left[b, b + A^{1/2}\right)_I$$

Scalar product

$$\langle f^* | g\rangle = (\pi d^3)^{-1/2} \int_S d^2 z \exp\left(-2z_I^2\right) f_T(z) g_T(z^*)$$

displacement operators

$$\begin{aligned} \mathcal{X} &= \exp[-\pi A^{-1/2} \partial_z] \\ \mathcal{P} &= \exp[i2\pi A^{-1/2} z - \pi^2 A^{-1}]\ \exp[i\pi A^{-1/2} \partial_z] \end{aligned}$$

- number of zeros

$$\oint_\ell \frac{dz}{2\pi i}\frac{\partial_z f_T(z)}{f_T(z)} = d$$

  exactly d zeros in the cell S with area πd.

  constraint for sum of zeros

$$\oint_\ell \frac{dz}{2\pi i}\frac{\partial_z f_T(z)}{f_T(z)} z = A^{1/2}\left[M + iN + \frac{d(1+i)}{2}\right]$$

  where M, N are integers.

- given d zeros ζ_n in the cell S **which obey the constraint**, the function $f_T(z)$ is:

$$\begin{aligned} f_T(z) &= C \exp\left[-i2\pi A^{-1/2} N z\right] \prod_n \Theta_3\left[w_n(z);\ i\right] \\ w_n(z) &= \pi A^{-1/2}(z - \zeta_n) + \frac{\pi(1+i)}{2} \end{aligned}$$

- zeros define uniquely the wavefunction
  (**not** in ∞-dim systems)
  how they evolve in time
  chaotic systems

Discussion

- general systems with discrete cyclic symmetries
  multivaluedness
  analytic repr. in the ℓ sheeted-complex plane

- Galois systems: Frobenius symmetry $\mathcal{G}^\ell = \mathbf{1}$
  and in Galois subsystems $\mathcal{G}^d \Pi_d = \Pi_d$ (for $d|\ell$)
  multivaluedness
  analytic repr. in the ℓ sheeted-complex plane

  links between harmonic analysis on Galois fields and Riemann surfaces

- Analytic repr. of d-dimensional systems with theta functions on a torus; their zeros

  links between theory of finite quantum systems and Theta functions

Other topics

D1. Spins: angle and angular momentum states

D2a. $SU(d)$ transformations

D2b. local and entangling transformations in bi-partite systems

D3. factorization of large systems in terms of smaller ones

Spins: angle and angular momentum states

- Apply theory of finite systems to spin j.
  dimension $d = 2j + 1$

  angular momentum states $|J; jm\rangle$

  $[J_z, J_+] = J_+; \quad [J_z, J_-] = -J_-; \quad [J_+, J_-] = 2J_z$

  Casimir operator

  $$J^2 = J_z^2 + \frac{1}{2}(J_+J_- + J_-J_+) = j(j+1)\mathbf{1}$$

  then

  $$\begin{aligned} J_+|J; j\ m\rangle &= [j(j+1) - m(m+1)]^{1/2}|J; j\ m+1\rangle \\ J_-|J; j\ m\rangle &= [j(j+1) - m(m-1)]^{1/2}|J; j\ m-1\rangle \\ J_z|J; j\ m\rangle &= m|J; j\ m\rangle \\ J^2|J; j\ m\rangle &= j(j+1)|J; j\ m\rangle \end{aligned}$$

- polar decomposition of J_+ and J_-

$$\begin{aligned} J_+ &= J_r X(1); \qquad J_- = [X(1)]^\dagger J_r \\ J_r &= (J_+ J_-)^{1/2}; \qquad X(1) = \sum_m |J; j\ m+1\rangle\langle J; j\ m| \end{aligned}$$

'radial operator' J_r
'exponential of the phase operator' $X(1)$

$$\begin{aligned} [J_r, J_z] &= 0 \\ J_r |J; j\ m\rangle &= [j(j+1) - m(m-1)]^{1/2} |J; j\ m\rangle \\ J_r &= [j(j+1)\mathbf{1} - J_z^2 + J_z]^{1/2} \end{aligned}$$

- Bose sector: integer j
Fermi sector: half-integer j
Here: Bose case

Fourier operator

$$F = (2j+1)^{-1/2} \sum_{m,n} \omega(mn) |J; j\ m\rangle\langle J; j\ n|$$

angle states

$$|\theta; j\ m\rangle = F|J; j\ m\rangle = (2j+1)^{-1/2} \sum_n \omega^{mn} |J; j\ n\rangle$$

angle operators

$$F J_z F^\dagger = \theta_z; \qquad F J_+ F^\dagger = \theta_+; \qquad F J_- F^\dagger = \theta_-$$

form the $SU(2)$ algebra

$$[\theta_z, \theta_+] = \theta_+; \quad [\theta_z, \theta_-] = -\theta_-; \quad [\theta_+, \theta_-] = 2\theta_z$$

Casimir operator

$$\theta^2 = \theta_z^2 + \frac{1}{2}(\theta_+\theta_- + \theta_-\theta_+) = j(j+1)\mathbf{1} = J^2$$

then

$$\begin{aligned} \theta_+|\theta; j\ m\rangle &= [j(j+1) - m(m+1)]^{1/2}|\theta; j\ m+1\rangle \\ \theta_-|\theta; j\ m\rangle &= [j(j+1) - m(m-1)]^{1/2}|\theta; j\ m-1\rangle \\ \theta_z|\theta; j\ m\rangle &= m|\theta; j\ m\rangle \\ \theta^2|\theta; j\ m\rangle &= j(j+1)|\theta; j\ m\rangle \end{aligned}$$

polar decomposition of θ_+ and θ_-

$$\begin{aligned} \theta_+ &= \theta_r Z(1); \quad \theta_- = [Z(1)]^\dagger \theta_r \\ \theta_r &= (\theta_+\theta_-)^{1/2} = F J_r F^\dagger \\ Z(1) &= F X(1) F^\dagger = \sum_m |\theta; j\ m+1\rangle\langle\theta; j\ m| \end{aligned}$$

‘radial operator’ θ_r
‘exponential of the phase operator’ $Z(1)$

- Heisenberg-Weyl group

  $X(d) = Z(d) = \mathbf{1}; \quad X(\beta)Z(\alpha) = Z(\alpha)X(\beta)\omega(-\alpha\beta)$

  where

$$\omega(\alpha) = \exp\left(i\frac{2\pi\alpha}{2j+1}\right)$$

  prove

$$\begin{aligned} Z(\alpha)|\theta; j\ m\rangle &= |\theta; j\ m+\alpha\rangle \\ Z(\alpha)|J; j\ m\rangle &= \omega(\alpha m)|J; j\ m\rangle \end{aligned}$$

$$\begin{aligned} X(\beta)|\theta; j\ m\rangle &= \omega(-m\beta)|\theta; j\ m\rangle \\ X(\beta)|J; j\ m\rangle &= |J; j\ m+\beta\rangle \end{aligned}$$

  also

$$X(\beta) = \omega(-\beta\theta_z); \qquad Z(\alpha) = \omega(\alpha J_z)$$

- Wigner and Weyl functions in this context

$SU(d)$ transformations:

$D(\alpha, \beta)$ generators of $SU(d)$ transformations

$d^2 - 1$generators ($D(0,0) = \mathbf{1}$)

- infinitesimal $SU(d)$ transformations:

$$g = \mathbf{1} + \sum_{\alpha,\beta} \epsilon(\alpha, \beta) D(\alpha, \beta)$$

- finite (non-infinitesimal) trans:

$$U = \sum_{\alpha,\beta} \mu(\alpha, \beta) D(\alpha, \beta)$$

$$\mu(\alpha, \beta) = \frac{1}{d} \tilde{W}(\Theta; -\alpha, -\beta)$$

- Finite transformations in terms of displaced parity operators:

$$U = \sum_{\alpha,\beta} \frac{1}{d} W(\Theta; \alpha, \beta) P(\alpha, \beta)$$

  The product of two parity operators is **not** a parity operator.

local and entangling transformations in bi-partite systems:

Hilbert space $\mathcal{H}_1 \times \mathcal{H}_2$

displacements: $D_1 = D \otimes \mathbf{1}$, $D_2 = \mathbf{1} \otimes D$.

- **local trans:** $SU(d_1) \times SU(d_2)$

  $D_1(\alpha_1, \beta_1)$ and $D_2(\alpha_2, \beta_2)$ generators

  $(d_1^2 - 1) + (d_2^2 - 1)$ generators

  infinitesimal local trans:

$$g = \mathbf{1} + \sum_{\alpha_1,\beta_1} \epsilon_1(\alpha_1, \beta_1) D_1(\alpha_1, \beta_1) + \sum_{\alpha_2,\beta_2} \epsilon_2(\alpha_2, \beta_2) D_2(\alpha_2, \beta_2)$$

  finite local trans:

$$U_1 \times U_2 = \frac{1}{d^2} \sum_{\alpha_1,\beta_1,\alpha_2,\beta_2} [\tilde{W}(U_1; -\alpha_1, -\beta_1) D_1(\alpha_1, \beta_1)] \times [\tilde{W}(U_2; -\alpha_2, -\beta_2) D_2(\alpha_2, \beta_2)]$$

- **local + entangling trans:** $SU(d_1 d_2)$

  generators **all** displacements $D_1(\alpha_1, \beta_1) D_2(\alpha_2, \beta_2)$

  $(d_1 d_2)^2 - 1$ generators: the local ones + entangling where **both** D_1 and D_2 non-trivial

  infinitesimal $SU(d_1 d_2)$ trans:

$$
\begin{aligned}
g = \mathbf{1} \quad & + \quad \sum_{\alpha_1, \beta_1} \epsilon_1(\alpha_1, \beta_1) D_1(\alpha_1, \beta_1) \\
& + \quad \sum_{\alpha_2, \beta_2} \epsilon_2(\alpha_2, \beta_2) D_2(\alpha_2, \beta_2)] \\
& + \quad \sum_{\alpha_1, \beta_1, \alpha_2, \beta_2} \epsilon_3(\alpha_1, \beta_1, \alpha_2, \beta_2) D_1(\alpha_1, \beta_1) D_2(\alpha_2, \beta_2)
\end{aligned}
$$

  finite $SU(d_1 d_2)$ trans:

$$
\begin{aligned}
U = \frac{1}{d^2} \sum_{\alpha_1, \beta_1, \alpha_2, \beta_2} & \tilde{W}(U; -\alpha_1, -\beta_1, -\alpha_2, -\beta_2) \\
& \times [D_1(\alpha_1, \beta_1) D_2(\alpha_2, \beta_2)]
\end{aligned}
$$

- $SU(d_1) \times SU(d_2)$ subgroup of $SU(d_1 d_2)$ (**not** a normal subgroup). $SU(d_1 d_2)/SU(d_1) \times SU(d_2)$ **not** a group.

- Symplectic transformations ($d_1 = d_2 = p$)

  local symplectic transformations:
  $Sp(2, \mathbb{Z}(p)) \times Sp(2, \mathbb{Z}(p))$

  local+ entangling symplectic transformations:

$$
\begin{aligned}
X'_i(1) &= S X_i(1) S^\dagger = [X_1(\alpha_{1i}) Z_1(\beta_{1i})][X_2(\alpha_{2i}) Z_2(\beta_{2i})] \\
Z'_i(1) &= S Z_i(1) S^\dagger = [X_1(\gamma_{1i}) Z_1(\delta_{1i})][X_2(\gamma_{2i}) Z_2(\delta_{2i})]
\end{aligned}
$$

  We require

$$
\begin{aligned}
&X'_i(p) = Z'_i(p) = \mathbf{1} \\
&X'_i(\beta) Z'_i(\alpha) = Z'_i(\alpha) X'_i(\beta) \omega(-\alpha\beta)
\end{aligned}
$$

  and

$$
\begin{aligned}
&[X'_1(1), X'_2(1)] = [X'_1(1), Z'_2(1)] \\
&= [Z'_1(1), X'_2(1)] = [Z'_1(1), Z'_2(1)] = 0
\end{aligned}
$$

  and get

$$
\begin{aligned}
(\alpha_{11}\beta_{12} - \beta_{11}\alpha_{12}) + (\alpha_{21}\beta_{22} - \beta_{21}\alpha_{22}) &= 0 \\
(\gamma_{11}\delta_{12} - \delta_{11}\gamma_{12}) + (\gamma_{21}\delta_{22} - \delta_{21}\gamma_{22}) &= 0 \\
(\alpha_{1i}\delta_{1k} - \beta_{1i}\gamma_{1k}) + (\alpha_{2i}\delta_{2k} - \beta_{2i}\gamma_{2k}) &= \delta(i,k)
\end{aligned}
$$

  16 integer parameters and 6 constraints.
  In the Galois case ‘solve’ the constraints

  10 **integer** parameter $Sp(4, \mathbb{Z}(p))$ group
  Find numerically operator S.

factorization of large systems in terms of smaller ones

- $d = d_1 \times ... \times d_N$ where d_1,...,d_n coprime
  two one-to-one mappings
  $$\mathbb{Z}(d) \ \leftrightarrow \ \mathbb{Z}(d_1) \times ... \times \mathbb{Z}(d_n)$$
  based on Chinese remainder theorem
  Good scheme for fast Fourier transf

- integers r_i and t_i:
  $$r_i = \frac{d}{d_i}; \qquad t_i r_i = 1 \ (mod \ d_i)$$
  t_i is the 'inverse' of r_i within $\mathbb{Z}(d_i)$
  exists because r_i and d_i are coprime.

  $s_i = t_i r_i$ in $\mathbb{Z}(d)$.

- first mapping
  $$m \leftrightarrow (m_1, ...m_N)$$
  $$m_i = m(mod \ d_i); \quad m = \sum_i m_i s_i$$
  second mapping
  $$m \leftrightarrow (\bar{m}_1, ..., \bar{m}_N)$$
  $$\bar{m}_i = m t_i (mod \ d_i); \quad m = \sum_i \bar{m}_i r_i (mod \ d)$$

- example $d = 15,\ d_1 = 3,\ d_2 = 5$
  $r_1 = 5,\ t_1 = 2,\ s_1 = 10$
  $r_2 = 3,\ t_2 = 2,\ s_2 = 6$

  $m = 10m_1 + 6m_2 = 5\bar{m}_1 + 3\bar{m}_2.$

  $m = 7 \ \leftrightarrow\ (m_1 = 1, m_2 = 2) \ \leftrightarrow\ (\bar{m}_1 = 2, \bar{m}_2 = 4)$

- Hilbert spaces

$$\mathcal{H}(d) \ \leftrightarrow\ \mathcal{H}(d_1) \otimes ... \otimes \mathcal{H}(d_N)$$

  with

$$|X; m\rangle \leftrightarrow |X_1; \bar{m}_1\rangle \otimes ... \otimes |X_N; \bar{m}_N\rangle$$
$$|P; m\rangle \leftrightarrow |P_1; m_1\rangle \otimes ... \otimes |P_N; m_N\rangle$$

-

$$D(\alpha, \beta) = \prod_{i=1}^{N} D_i(\alpha_i, \bar{\beta}_i)$$
$$P(\alpha, \beta) = \prod_{i=1}^{N} P_i(\alpha_i, \bar{\beta}_i)$$

- density matrix Θ with matrix elements: $|X;m\rangle$ and also $|P;m\rangle$:

  $\sigma(n,m) \equiv \langle X;n|\Theta|X;m\rangle; \qquad \tau(\ell,k) \equiv \langle P;\ell|\Theta|P;k\rangle$

  $\sigma(\{\bar{n}_i\},\{\bar{m}_j\})$ matrix elements of Θ in the basis $|X_1;\bar{m}_1\rangle \otimes ... \otimes |X_N;\bar{m}_N\rangle$

  $\tau(\{\ell_i\},\{k_j\})$ matrix elements of Θ in the basis $|P_1;m_1\rangle \otimes ... \otimes |P_N;m_N\rangle$

  $\sigma(n,m)$ and $\tau(\ell,k)$ related as:

  $$\sigma(n,m) = d^{-1}\sum_{\ell,k}\tau(\ell,k)\omega(\ell n - mk)$$

  'large' Fourier transform equivalent to the combination of 'small' Fourier transforms

  $$\sigma(\{\bar{n}_i\},\{\bar{m}_j\}) = \left(\prod_{i=1}^{N} d_i^{-1}\right)\sum_{\ell_i,k_i}\tau(\{\ell_i\},\{k_j\}) \prod_{i=1}^{N}\omega_i(\ell_i\bar{n}_i - \bar{m}_i k_i)$$

  used in 'fast Fourier transforms'.

- Wigner and Weyl functions

$$W(\Theta;\alpha,\beta) = W(\Theta;\{\alpha_i\},\{\bar{\beta}_i\})$$
$$\tilde{W}(\Theta;\alpha,\beta) = \tilde{W}(\Theta;\{\alpha_i\},\{\bar{\beta}_i\})$$

- $SU(d)$ trans:

$$\begin{aligned} U &= \frac{1}{d}\sum_{\alpha,\beta} \tilde{W}\left(U;-\alpha,-\beta\right) D(\alpha,\beta) \\ &= \frac{1}{d}\sum_{\{\alpha_i\},\{\beta_i\}} \tilde{W}\left(U;\{-\alpha_i\},\{-\bar{\beta}_i\}\right) \prod_{i=1}^{N} D_i(\alpha_i,\bar{\beta}_i) \end{aligned}$$

trans in the subgroup $SU(d_1)\times ... \times SU(d_N)$ of $SU(d)$

$$U = \frac{1}{d}\prod_{i=1}^{N}\left(\sum_{\alpha_i,\bar{\beta}_i} \tilde{W}(U_i;-\alpha_i,-\bar{\beta}_i) D_i(\alpha_i,\bar{\beta}_i)\right)$$

Discussion

- apply theory of finite systems to spins
  angle states dual to angular momentum states
  angle operators ($SU(2)$ group)

- displacement operators as generators of unitary $SU(d)$ trans

  bipartite systems
  $SU(d_1 d_2)$ local and entangling trans
  its subgroup $SU(d_1) \times SU(d_2)$ local trans

- factorization of a large system in terms of smaller ones
  similar to Good's scheme in fast Fourier trans
  uses Chinese remainder theorem