A COMPARATIVE STUDY BETWEEN BIHARMONIC BÉZIER SURFACES AND BIHARMONIC EXTREMAL SURFACES

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ABSTRACT

Given a prescribed boundary of a Bézier surface we compare the Bézier surfaces generated by two different methods, i.e. the Bézier surface minimising the Biharmonic functional and the unique Bézier surface solution of the Biharmonic equation with prescribed boundary. Although often the two types of surfaces look visually the same, we show that they are indeed different. In this paper we provide a theoretical argument showing why the two types of surfaces are not always the same.

KEY WORDS

Bézier surfaces, Biharmonic equation, Biharmonic functional

1 Introduction

In many areas of science and engineering the related physical phenomena can be modelled using fourth order linear elliptic Partial Differential Equations (PDEs). In this paper, what we are addressing falls within the general setting of geometric modelling particularly the generation of geometric surfaces by means of solutions to elliptic PDEs. The problem we are addressing here can be described as follows. Given four boundary curves find a parametric surface patch $\overrightarrow{\mathbf{x}}$ such that $\overrightarrow{\mathbf{x}} : [u, v] \to \mathbf{R}^3$ whereby the surface patch $\overrightarrow{\mathbf{x}}$ smoothly interpolates the four curves. We assume the four boundary curves are defined as $\overrightarrow{\mathbf{x}}(u, 0)$, $\overrightarrow{\mathbf{x}}(u, 1)$, $\overrightarrow{\mathbf{x}}(v, 0)$ and $\overrightarrow{\mathbf{x}}(v, 1)$ where the domain of $\overrightarrow{\mathbf{x}}$ is the unit square $0 \leq u, v \leq 1$. In this particular case for the smooth interpolation we solve an elliptic PDE subject to four boundary conditions at the edges of the surface patch. Here the solution of the PDE is expressed as a polynomial function commonly known as a polynomial surface patch. Such polynomial surfaces are common in the area of Computer-Aided Geometric Design and examples include Bézier surfaces and B-Splines[4].

Given a parametric surface $\overrightarrow{\mathbf{x}} : [0, 1]^2 \to \mathbf{R}^3$, the surface is said to be a Biharmonic Bézier surface if for a given set of boundary conditions the Bézier polynomial function describing the surface satisfies $\Delta^2 \overrightarrow{\mathbf{x}} = 0$ where,
$\Delta \overrightarrow{\mathbf{x}} = \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) \overrightarrow{\mathbf{x}} = \overrightarrow{\mathbf{x}}_{uu} + \overrightarrow{\mathbf{x}}_{vv}$.

Our broad aim in working with Bézier surfaces satisfying elliptic boundary value problems such as the Biharmonic equation is to develop boundary based intuitive surface design techniques for polynomial surfaces. By means of such techniques the shape of the resulting polynomial surface can be easily manipulated through its boundaries. We are particularly working with Bézier surfaces since they are one of the basic types of surfaces widely used in Computer-Aided Geometric Design[4].

Our work on this theme is similar to the work based on variational approaches to geometric modelling formulations [3, 11, 6]. Thus, we associate the polynomial functions of Bézier surfaces with geometric boundary-value problems, in particular elliptic Partial Differential Equations (PDEs). This enables us to generate Bézier surfaces with a wide variety of desired properties. For example, choosing a Bézier surface which verifies a given boundary-value problem such as the standard Biharmonic PDE enables us to generate a surface which be can solely controlled through the boundary control points. Note: in the classical case, surfaces based on polynomial functions involve control points which are spread to the entire surface. Manipulation of such surfaces through the direct manipulation of control points is often not intuitive.

In this paper we discuss the similarities and differences between the Biharmonic Bézier surfaces, (i.e. Bézier polynomial solutions of the Biharmonic equation) and the corresponding Biharmonic extremal surfaces, (i.e. extremals of the Biharmonic functional among all polynomial patches of a given degree with the same prescribed boundary). In our earlier work discussed in [9] some of the experimental results showed that for a given set of boundary conditions a Biharmonic Bézier surface and a corresponding Biharmonic extremal surface, although they often visually look the same, they are indeed different. However, we did not provide a concrete proof with an explanation of this result.

In this paper we provide a theoretical argument showing why the two types of surfaces are not always the same. Furthermore, we provide the necessary and sufficient conditions for the two types of surfaces to agree.

There exist literature on methods for generating Bézier surfaces verifying elliptic boundary value problems, in particular for boundary value problems associated with the Laplace equation as well as the Biharmonic equation which are referred to as Harmonic and Biharmonic Bézier surfaces respectively [9]. The main point we note from this previous work is that both the Harmonic and Biharmonic Bézier surfaces are related to minimal surfaces. i.e. surfaces that minimise the area among all the surfaces with prescribed boundary data. In the Harmonic case two boundary conditions are required to construct the surface. Similarly in the Biharmonic case four boundary conditions are required to satisfy the fourth order elliptic PDE. It is also important to highlight that for the Biharmonic case, even though the chosen boundary-value problem is of fourth order, the knowledge of the boundaries defining the edges of surface patch alone enables one to fully determine the entire surface. This is mainly due to the fact that we are looking for polynomial solutions of the associated PDE. Detailed discussions of Harmonic Bézier surfaces and Biharmonic Bézier surfaces can be found in [8, 9].

When one is concerned with generating surfaces conforming to boundaries, it is common practise to look for a surface satisfying the extremal of a functional among all surfaces with the same prescribed boundary [10]. Various basic functionals can be utilised for this purpose. For example,

$$\int_{[0,1]^2} ||\overrightarrow{\mathbf{x}}_u||^2 + ||\overrightarrow{\mathbf{x}}_v||^2 du dv,$$

corresponds to the Harmonic functional, or Dirichlet functional. Another typical example is,

$$\int_{[0,1]^2} ||\overrightarrow{\mathbf{x}}_{uv}||^2 du dv,$$

corresponding to the Coons functional. Yet another example is,

$$\int_{[0,1]^2} ||\overrightarrow{\mathbf{x}}_{uu}||^2 + 2||\overrightarrow{\mathbf{x}}_{uv}||^2 + ||\overrightarrow{\mathbf{x}}_{vv}||^2 dudv,$$

which corresponds to the Biharmonic functional. Apart from the above common functionals, other higher order functionals, or functionals with added terms or with modifying parameters also have been utilised.

Recently the idea of extremal surfaces has been utilised for a variety of smooth surface construction applications in geometric modelling. For example, Guy and Medioni [5] and Medioni et. al. [7] have utilised extremal surfaces for surface reconstruction from noisy point clouds. Adamson and Alexa [1] discuss implicit surfaces associated with extremal surfaces which they utilise for ray-tracing purposes. Other variations of extremal surfaces such as Moving Least Square surfaces or more widely known as MLS surfaces have been studied by Amenta and Kil [2].

It is well known that the extremals of the above mentioned functionals are solutions of the corresponding Euler-Lagrange equation. i.e. the Euler-Lagrange equations corresponding to the Harmonic, Coons and Biharmonic functionals are the Harmonic ($\Delta \overrightarrow{\mathbf{x}} = 0$), Coons ($\overrightarrow{\mathbf{x}}_{uuvv} = 0$) and Biharmonic equation ($\Delta^2 \overrightarrow{\mathbf{x}} = 0$) respectively. This is true for the unrestricted case, i.e., the extremal of the functional among all smooth patches $\overrightarrow{\mathbf{x}} \in C^\infty([0,1]^2)$. Note that, in general, the solution of the Euler-Lagrange equation does not need to be a polynomial function, even if the boundary is a polynomial function. For a given boundary here we are concerned with solutions of Biharmonic PDEs and the extremal of the corresponding Euler-Lagrange equation where the solutions conform to polynomial patches of a given degree.

Considering the functional we are interested in is quadratic and of second order, the corresponding Euler-Lagrange equation is a linear fourth-order PDE. Thus, there are two different ways to generate a Bézier surface with prescribed boundary, i.e. to look for the extremal of a second order quadratic functional among all polynomial patches or to look for the unique polynomial solution of the corresponding Euler-Lagrange equation. However, as mentioned earlier, the two kinds of surfaces are not necessarily the same. In fact we show that the only one case where extremals and solutions agree is the Coons case and in general the extremals and PDE solutions are different.

The paper is organised as follows. In the next section we describe our main result in the form of a theorem and its proof describing the necessary and sufficient conditions for a Bézier polynomial solution satisfying the Biharmonic equation to be an extremal of the Biharmonic functional. In Section 3 we discuss some examples whereby we highlight the similarities and differences between the Biharmonic Bézier surfaces and the corresponding Biharmonic extremal surfaces. Finally in Section 4 we conclude the paper.

2 The argument

In this section we discuss the main result of the paper. In particular, we discuss the necessary and sufficient conditions for a Biharmonic Bézier surface to be the same as the corresponding Biharmonic extremal surface.

Given two functions $f, g : [0,1] \to \mathbf{R}$, in the space of square integrable functions defined on $[0,1]$ their scalar product is defined by means of the integral such that,
$< f, g >= \int_0^1 f(t)g(t)dt.$
The pair of functions are orthogonal if $< f, g >= 0$.
Lemma: Let f be a polynomial function of degree $n -$

2. If f is orthogonal to all polynomial curves of degree n vanishing at 0 and 1, then $f \equiv 0$.

Proof: We will assume that if a polynomial function of degree n is orthogonal to any other polynomial function of degree n, then it is the null polynomial.

Let us write $B_i = t^{i-1}(1-t)$ with $i \in \{2, \dots, n\}$. The family $\{B_i\}_{i=2}^{n}$ is a basis of the subspace of polynomial functions of degree n vanishing at 0 and 1. Note that the degree of B_i is i.

The family $\{B_2, B_3, \dots, B_n, 1, t\}$ is a basis of the vector space of degree n polynomials. If we apply the Gramm-Schmidt orthonormalization process to the basis $\{B_2, B_3, \dots, B_n, 1, t\}$ (the order is important) we get an orthonormal basis $\{L_2, L_3, \dots, L_n, H_0, H_1\}$. Consequences of the Gramm-Schmidt process are that $\{L_2, L_3, \dots, L_n\}$ is an orthonormal basis of polynomial functions of degree n vanishing at 0 and 1, the degree of L_i is i and the degrees of H_0 and H_1 are both n.

Now, let f be a polynomial function of degree $n-2$ orthogonal to all polynomial functions of degree n vanishing at 0 and 1. It is easy to check that $f - < f, H_0 > H_0 - < f, H_1 > H_1$ is orthogonal to any other degree n polynomial. Indeed it is orthogonal to any element of the orthogonal basis. Therefore it must be null. Since the degrees of H_0 and H_1 are both n whereas the degree of f is $n-2$, then $f - < f, H_0 > H_0 - < f, H_1 > H_1 = 0$ implies $f = 0$.

Theorem: Let $\overrightarrow{\mathbf{y}} : [0,1] \times [0,1] \to \mathbf{R}^3$ be a polynomial solution of degree $n \geq 3$ of the Biharmonic equation, then $\overrightarrow{\mathbf{y}}$ is also the extremal of the Biharmonic functional among all polynomial patches of the same degree with the same boundary if and only if the transversal second partial derivatives, $\overrightarrow{\mathbf{y}}_{uu}(0,v)$, $\overrightarrow{\mathbf{y}}_{uu}(1,v)$, $\overrightarrow{\mathbf{y}}_{vv}(u,0)$ and $\overrightarrow{\mathbf{y}}_{vv}(u,1)$, along the four boundary curves vanish. For the case $n=2$, the condition is such that the sum of the four corner control points agree with the sum of the other four boundary control points $P_{0,0} + P_{2,0} + P_{0,2} + P_{2,2} = P_{1,0} + P_{0,1} + P_{1,2} + P_{2,1}$.

Proof: The extremal, $\overrightarrow{\mathbf{y}}$, of a functional $\mathcal{F}$ is characterised by, $\frac{d}{dt}|_{t=0}\mathcal{F}(\overrightarrow{\mathbf{y}} + t\overrightarrow{\mathbf{z}}) = 0$, for any polynomial patch of degree n, $\overrightarrow{\mathbf{z}} : [0,1] \times [0,1] \to \mathbf{R}^3$, vanishing at the boundary.

For the biharmonic functional this is equivalent to,

$$\begin{aligned} 0 &= \int_{[0,1]^2} (< \overrightarrow{\mathbf{y}}_{uu}, \overrightarrow{\mathbf{z}}_{uu} > +2 < \overrightarrow{\mathbf{y}}_{uv}, \overrightarrow{\mathbf{z}}_{uv} > + \\ & < \overrightarrow{\mathbf{y}}_{vv}, \overrightarrow{\mathbf{z}}_{vv} >) du dv. \end{aligned}$$

By integration by parts,

$$\begin{aligned} 0 &= \int_{[0,1]^2} ((< \overrightarrow{\mathbf{y}}_{uu}, \overrightarrow{\mathbf{z}}_{u} >)_u + 2(< \overrightarrow{\mathbf{y}}_{uv}, \overrightarrow{\mathbf{z}}_{u} >)_v + \\ & (< \overrightarrow{\mathbf{y}}_{vv}, \overrightarrow{\mathbf{z}}_{v} >)_v) du dv - \int_{[0,1]^2} (< \overrightarrow{\mathbf{y}}_{uuu}, \overrightarrow{\mathbf{z}}_{u} > \\ & +2 < \overrightarrow{\mathbf{y}}_{uvv}, \overrightarrow{\mathbf{z}}_{u} > + < \overrightarrow{\mathbf{y}}_{vvv}, \overrightarrow{\mathbf{z}}_{v} >) du dv \\ &= \int_0^1 < \overrightarrow{\mathbf{y}}_{uu}(1,v), \overrightarrow{\mathbf{z}}_{u}(1,v) > dv - \\ & \int_0^1 < \overrightarrow{\mathbf{y}}_{uu}(0,v), \overrightarrow{\mathbf{z}}_{u}(0,v) > dv \\ & +2 \int_0^1 < \overrightarrow{\mathbf{y}}_{uv}(u,1), \overrightarrow{\mathbf{z}}_{u}(u,1) > du \\ & -2 \int_0^1 < \overrightarrow{\mathbf{y}}_{uv}(u,0), \overrightarrow{\mathbf{z}}_{u}(u,0) > du \\ & + \int_0^1 < \overrightarrow{\mathbf{y}}_{vv}(u,1), \overrightarrow{\mathbf{z}}_{v}(u,1) > du \\ & - \int_0^1 < \overrightarrow{\mathbf{y}}_{vv}(u,0), \overrightarrow{\mathbf{z}}_{v}(u,0) > du \\ & + \int_{[0,1]^2} (< \overrightarrow{\mathbf{y}}_{uuuu}, \overrightarrow{\mathbf{z}} > +2 < \overrightarrow{\mathbf{y}}_{uuvv}, \overrightarrow{\mathbf{z}} > \\ & + < \overrightarrow{\mathbf{y}}_{vvvv}, \overrightarrow{\mathbf{z}} >) du dv. \end{aligned}$$

The last term is nothing but, $\int_{[0,1]^2} < \Delta^2 \overrightarrow{\mathbf{y}}, \overrightarrow{\mathbf{z}} > du dv$, which vanishes because, by definition, $\overrightarrow{\mathbf{y}}$ is a solution of the biharmonic equation.

Moreover, note that since $\overrightarrow{\mathbf{z}}$ vanishes along the boundaries, then $\overrightarrow{\mathbf{z}}(u,1) = 0$, and then the factor $\overrightarrow{\mathbf{z}}_u(u,1)$ also vanishes. Analogously $\overrightarrow{\mathbf{z}}_u(u,0)$, $\overrightarrow{\mathbf{z}}_v(1,v)$ and $\overrightarrow{\mathbf{z}}_v(0,v)$ also vanish along the boundaries.

Therefore, $\overrightarrow{\mathbf{y}}$ is the extremal of the biharmonic functional if and only if, for any $\overrightarrow{\mathbf{z}}$ vanishing at the boundaries, the following expression vanishes.

$$\begin{aligned}
&\int_0^1 < \overrightarrow{\mathbf{y}}_{uu}(1,v), \overrightarrow{\mathbf{z}}_u(1,v) > dv \\
&- \int_0^1 < \overrightarrow{\mathbf{y}}_{uu}(0,v), \overrightarrow{\mathbf{z}}_u(0,v) > dv \\
&+ \int_0^1 < \overrightarrow{\mathbf{y}}_{vv}(u,1), \overrightarrow{\mathbf{z}}_v(u,1) > du \\
&- \int_0^1 < \overrightarrow{\mathbf{y}}_{vv}(u,0), \overrightarrow{\mathbf{z}}_v(u,0) > du.
\end{aligned} \tag{1}$$

Such a $\overrightarrow{\mathbf{z}}$ must be a linear combination of the products of the Bernstein polynomials vanishing at 0 and 1, $\{B_i^n(u)B_j^n(v)\}_{i,j=1}^{n-1}$. To justify this we can simply use the properties of Bernstein polynomials [4]. The 'basis property' of Bernstein polynomials states that any polynomial of degree n can be uniquely written as a linear combination of the Bernstein polynomials of order n. i.e. Bernstein polynomials form a basis of the space of all polynomials of degree n. Let $P(t)$ a polynomial of degree n on t. Since the family of n-degree Bernstein polynomials $\{B_i^n(t)\}_{i=0}^n$ is a basis of polynomials of degree n, then $P(t) = \sum_{i=0}^n B_i^n(t)Q_i$ for some coefficients $Q_0, Q_1, \ldots, Q_n$. Furthermore, the 'interval end conditions property' of Bernstein polynomials states that $B_i^n(0) = \delta_i^0$, and $B_i^n(1) = \delta_i^n$, for all $i \in n$ where δ_j^i is the Kronecker's delta function. Therefore $P(0) = Q_0$ and $P(1) = Q_n$. If polynomial $P(t)$ vanishes at the boundary of $[0,1]$ then $Q_0 = Q_n = 0$ and $P(t)$ is just a linear combination of the Bernstein polynomials $\{B_i^n(t)\}_{i=1}^{n-1}$, those Bernstein polynomials vanishing at the boundary.

Now, if $\overrightarrow{\mathbf{z}} = \sum_{i,j=1}^{n-1} B_i^n(u)B_j^n(v)Q_{i,j}$, with $Q_{i,j} \in \mathbf{R}^3$, then, from the properties of the derivatives of Bernstein polynomials [4],

$$\begin{aligned}
\overrightarrow{\mathbf{z}}_u(1,v) &= \textstyle\sum_{j=1}^{n-1} B_j^n(v)Q_{n-1,j}, \\
\overrightarrow{\mathbf{z}}_u(0,v) &= -\textstyle\sum_{j=1}^{n-1} B_j^n(v)Q_{1,j}, \\
\overrightarrow{\mathbf{z}}_v(u,1) &= \textstyle\sum_{i=1}^{n-1} B_i^n(u)Q_{i,n-1}, \\
\overrightarrow{\mathbf{z}}_v(u,0) &= -\textstyle\sum_{i=1}^{n-1} B_i^n(u)Q_{i,1}.
\end{aligned}$$

Therefore, Expression (1) can be written as,

$$\begin{aligned}
&\textstyle\sum_{j=1}^{n-1}(\int_0^1 < \overrightarrow{\mathbf{y}}_{uu}(1,v), B_j^n(v) > dv)Q_{n-1,j} \\
&+\textstyle\sum_{j=1}^{n-1}(\int_0^1 < \overrightarrow{\mathbf{y}}_{uu}(0,v), B_j^n(v) > dv)Q_{1,j} \\
&+\textstyle\sum_{i=1}^{n-1}(\int_0^1 < \overrightarrow{\mathbf{y}}_{vv}(u,1), B_i^n(u) > du)Q_{i,n-1} \\
&+\textstyle\sum_{i=1}^{n-1}(\int_0^1 < \overrightarrow{\mathbf{y}}_{vv}(u,0), B_i^n(u) > du)Q_{i,1}.
\end{aligned}$$

Since the points $\{Q_{i,j}\}_{i,j=1}^{n-1}$ are arbitrary, then it can now be deduced that, for $n \geq 3$, the next four integrals,

$$\begin{aligned}
&\int_0^1 < \overrightarrow{\mathbf{y}}_{uu}(1,v), B_j^n(v)e_k > dv, \\
&\int_0^1 < \overrightarrow{\mathbf{y}}_{uu}(0,v), B_j^n(v)e_k > dv, \\
&\int_0^1 < \overrightarrow{\mathbf{y}}_{vv}(u,1), B_i^n(u)e_k > du, \\
&\int_0^1 < \overrightarrow{\mathbf{y}}_{vv}(u,0), B_i^n(u)e_k > du,
\end{aligned}$$

vanish for any $i,j \in \{1,\ldots n-1\}$ and $k \in \{1,2,3\}$, where $e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)$. This is equivalent to saying that the curves, $\overrightarrow{\mathbf{y}}_{uu}(0,v), \overrightarrow{\mathbf{y}}_{uu}(1,v), \overrightarrow{\mathbf{y}}_{vv}(u,0), \overrightarrow{\mathbf{y}}_{vv}(u,1), \quad u,v \in [0,1]$, are orthogonal to polynomial curves of degree n vanishing at 0 and 1. Now, from the Lemma, the four curves are identically zero.

For the case $n = 2$, the set $\{Q_{i,j}\}_{i,j=1}^{n-1}$ is reduced to just a point $\{Q_{1,1}\}$, and then the condition reads,

$$0 = \int_0^1 < \overrightarrow{\mathbf{y}}_{uu}(1,v), B_1^2(v)Q_{1,1} > dv$$
$$+ \int_0^1 < \overrightarrow{\mathbf{y}}_{uu}(0,v), B_1^2(v)Q_{1,1} > dv$$
$$+ \int_0^1 < \overrightarrow{\mathbf{y}}_{vv}(u,1), B_1^2(u)Q_{1,1} > du$$
$$+ \int_0^1 < \overrightarrow{\mathbf{y}}_{vv}(u,0), B_1^2(u)Q_{1,1} > du,$$
$$= \int_0^1 < \overrightarrow{\mathbf{y}}_{uu}(1,t) + \overrightarrow{\mathbf{y}}_{uu}(0,t)$$
$$+ \overrightarrow{\mathbf{y}}_{vv}(t,1) + \overrightarrow{\mathbf{y}}_{vv}(t,0), B_1^2(t)Q_{1,1} > dt.$$

From the Lemma the sum of the second derivatives of the four boundary curves vanishes. Since the four boundary curves are of degree 2, then their second derivatives are just constant functions. In fact,

$$\overrightarrow{\mathbf{y}}_{uu}(1,t) + \overrightarrow{\mathbf{y}}_{uu}(0,t) + \overrightarrow{\mathbf{y}}_{vv}(t,1) + \overrightarrow{\mathbf{y}}_{vv}(t,0)$$
$$= P_{2,2} - 2P_{1,2} + P_{0,2} + P_{2,0} - 2P_{1,0} + P_{0,0}$$
$$+ P_{2,2} - 2P_{2,1} + P_{2,0} + P_{0,2} - 2P_{0,1} + P_{0,0},$$

and the statement results.

We know that given a polynomial boundary of degree n there exists a unique solution of the Biharmonic equation with that boundary. The conditions stated in the theorem say that not any boundary configuration will give us solutions of the Biharmonic equation and at the same time extremals of the Biharmonic functional.

A point noteworthy here, which comes as a bi-product the above theorem, is that for case $n = 3$ the boundaries of the surface must be straight segments and the surface should form a hyperbolic paraboloid defined by the four corner boundary control points, in order for the surface to be Biharmonic and at the same time be an extremal of the Biharmonic functional.

Another point noteworthy is that the transversal second partial derivative, $\overrightarrow{\mathbf{y}}_{uu}(0,v)$ vanishes if and only if $P_{2,j} - 2P_{1,j} + P_{0,j} = 0$ for all $j = 0, \ldots, n$, or equivalently, if and only if the three control points $P_{2,j}, P_{1,j}$ and $P_{0,j}$ are collinear.

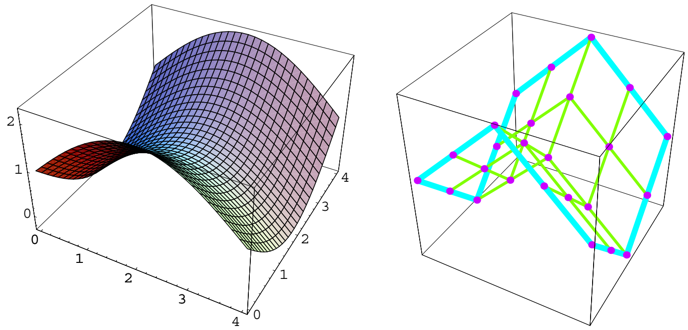

Figure 1. Left: A Biharmonic Bézier surface of degree 4 which is also the extremal of the Biharmonic functional for the associated boundary curves. Right: The control net showing the conditions in the statement of Theorem are satisfied, i.e. the coordinate lines have a vanishing transversal second partial derivative at the boundaries or equivalently, the first and last three control points for any coordinate line are collinear.

3 Examples

In this section we discuss some examples highlighting the similarities as well as the differences between the Biharmonic Bézier surfaces and the corresponding Biharmonic extremal surfaces.

3.1 Example 1

As a first example, we show a surface satisfying the Biharmonic equation verifying the extremal of the Biharmonic functional. We consider the Bézier surface of degree $n = 4$ defined as, $\overrightarrow{\mathbf{x}}(u,v) = (4u, 4v, 1 + 4u - 8u^3 + 4u^4 - 4v + 8v^3 - 4v^4)$, as shown in Figure 1. It is straightforward to check that this surface is indeed Biharmonic. Moreover, both $\overrightarrow{\mathbf{x}}_{uu}(u,v) = (0,0,48u(u-1))$, at $u = 0$ or $u = 1$ and $\overrightarrow{\mathbf{x}}_{vv}(u,v) = (0,0,48v(1-v))$, at $v = 0$ or $v = 1$ vanishes. i.e. the transversal derivatives of the four boundary curves vanish. Thus, according to the Theorem, the Bézier surface is also the extremal of the Biharmonic functional for the associated boundary conditions.

3.2 Example 2

In this example, we consider the following boundary control points of a degree 2 Bézier surface,

$$\begin{aligned} P_{00} &= (0,0,1), \\ P_{10} &= (1,0,0), \\ P_{20} &= (2,0,1), \\ P_{01} &= (0,1,0), \\ P_{21} &= (2,1,0), \\ P_{02} &= (0,2,1), \\ P_{12} &= (1,2,0), \\ P_{22} &= (2,2,1). \end{aligned}$$

The unique solution of the Biharmonic equation is obtained for the inner control point, $P_{1,1} = (1,1,-1)$. The associated Bézier surface is,

$$\overrightarrow{\mathbf{y}}(u,v) = (2v, 2u, 1 - 2(1-u)u - 2(1-v)v).$$

The value of the Biharmonic functional at $\overrightarrow{\mathbf{y}}$ is 32. However, the extremal of the Biharmonic functional with the same boundary is obtained for $P_{1,1} = (1,1,\frac{4}{11})$. The associated Bézier surface is,

$$\overrightarrow{\mathbf{x}}(u,v) = (2v, 2u, 1 - 2(1-v)v$$
$$-2u(1 + \tfrac{30}{11}(1-v)v) + u^2(2 - \tfrac{60}{11}(1-v)v))).$$

.

Figure 2 shows the Bézier surfaces for the Biharmonic equation and that for the Biharmonic extremal.

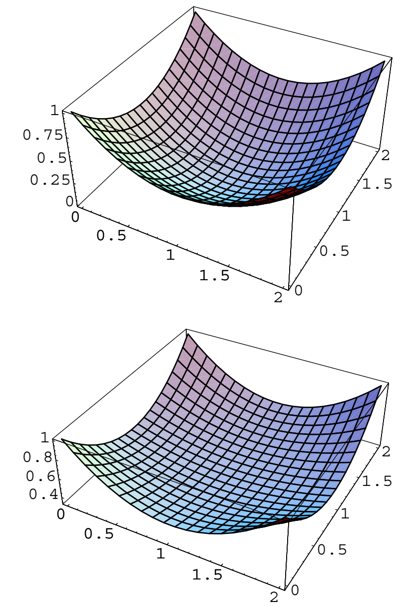

Figure 2. Top: The Biharmonic solution. Bottom: The Biharmonic extremal.

polynomial degree	functional value at the extremal
2	17.454545
4	16.114127
6	16.027878
8	16.010064

Table 1. Comparison of polynomial degree with the Biharmonic functional at the extremal.

Here we note that the conditions in the statement of the theorem are not verified. Indeed, the sum of the four corner points is $P_{0,0}+P_{2,0}+P_{0,2}+P_{2,2} = (4,4,4)$ whereas the sum of the other four boundary control points is $P_{1,0} + P_{0,1} + P_{1,2} + P_{2,1} = (4,4,0)$.

With the same boundary, but after a degree raising process, extremals of the Biharmonic functional for higher degrees can also be computed. Table 1 shows a list of the values of the functional at the extremals for different degrees.

3.3 Example 3

As a bi-product of the proof of the previous result we have shown that when the curvature of the coordinate lines transversal to the boundary curves is high then there are substantial differences between the Biharmonic solution and the Biharmonic extremal. The following example illustrates this.

Let us consider the following boundary control points of a degree 3 Bézier surface.

$$\begin{aligned} P_{00} &= (0,0,0), \\ P_{10} &= (\tfrac{3}{2},0,-3), \\ P_{20} &= (\tfrac{3}{2},0,3), \\ P_{30} &= (3,0,0), \\ P_{01} &= (0,\tfrac{3}{2},-3), \\ P_{31} &= (3,1,1), \\ P_{02} &= (0,\tfrac{3}{2},3), \\ P_{32} &= (3,2,1), \\ P_{03} &= (0,3,0), \\ P_{13} &= (1,3,-1), \\ P_{23} &= (2,3,-1), \\ P_{33} &= (3,3,0). \end{aligned}$$

The inner control points (i.e. all the control points accept the boundary control points) of the extremal of the Biharmonic functional are,

$$\begin{aligned} P_{11} &= (\tfrac{424}{493},\tfrac{424}{493},0), \\ P_{12} &= (\tfrac{1055}{493},\tfrac{511}{493},-\tfrac{32}{87}), \\ P_{21} &= (\tfrac{511}{493},\tfrac{1055}{493},\tfrac{32}{87}), \\ P_{22} &= (\tfrac{968}{493},\tfrac{968}{493},0), \end{aligned}$$

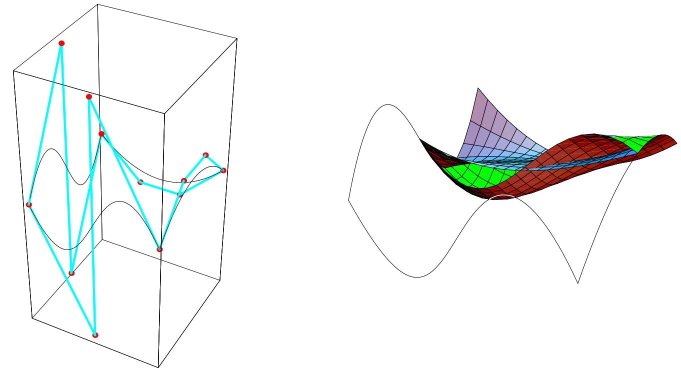

Figure 3. Left: The boundary control points and the common boundary curves. Right: Partial view of the Biharmonic extremal and the Biharmonic solution showing their different shapes.

and the value of the functional at the extremal is 453.072.

On the other hand, the inner control points of the Biharmonic solution are,

$$\begin{aligned} P_{11} &= (\tfrac{4}{3},\tfrac{4}{3},0), \\ P_{12} &= (\tfrac{5}{3},\tfrac{7}{6},\tfrac{10}{3}), \\ P_{21} &= (\tfrac{7}{6},\tfrac{5}{3},-\tfrac{10}{3}), \\ P_{22} &= (\tfrac{11}{6},\tfrac{11}{6},0), \end{aligned}$$

and the value of the functional at the extremal is 768.6. Figure 3 shows the surfaces resulting from the Biharmonic extremal and the Biharmonic solution along with the boundary control points and the common boundary curves. As one would notice the extremal tends to be more planar whilst the Biharmonic solution tends to reproduce the shapes of the boundary curves.

4 Conclusions

In this work we study the Biharmonic Bézier surfaces in comparison with the method of functional minimisations. We compare Biharmonic Bézier surfaces and those generated as an extremal of the Biharmonic functional. In particular we show that for given boundary conditions the Biharmonic surface and the resulting extremal surface in general are different except when the transversal second partial derivatives for the four boundary curves vanish.

As bi-products of the main result we presented in the paper, for the case $n = 2$, the condition is such that the sum of the four corner control points agree with the sum of the other four boundary control points. Moreover, for case $n = 3$, the surface must be a hyperbolic paraboloid defined by the four corner boundary control points in order for the surface to be Biharmonic and at the same time be an extremal of the Biharmonic functional.

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