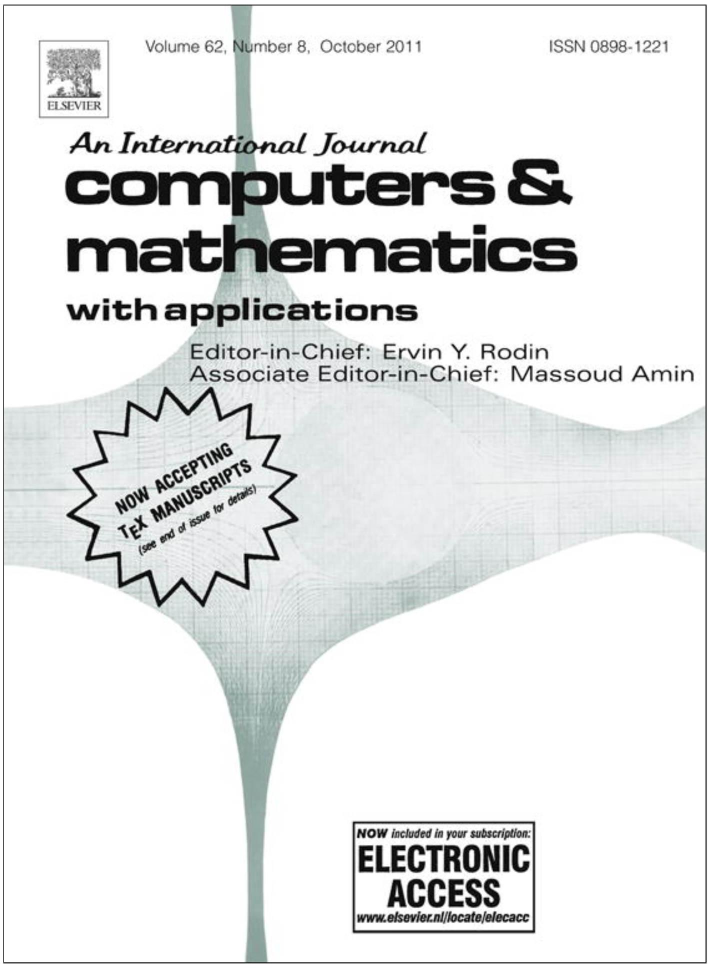
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On Bézier surfaces in three-dimensional Minkowski space

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ABSTRACT

In this paper, we study Bézier surfaces in $\mathbb{R}^3_1$ three-dimensional Minkowski space. In particular, we focus on timelike and spacelike cases for Bézier surfaces. We also deal with the Plateau–Bézier problem in $\mathbb{R}^3_1$, obtaining conditions over the control net to be extremal of the Dirichlet function for both timelike and spacelike Bézier surfaces. Moreover, we provide interesting examples showing the behavior of the Plateau–Bézier problem in $\mathbb{R}^3_1$ and illustrating the relationship between it and the corresponding Plateau–Bézier problem in the Euclidean space $\mathbb{R}^3$.

0. Introduction

A Bézier surface is defined using mathematical spline functions whereby the resulting surface has a compact analytic description. This enables such surfaces to be easily manipulated, and they also have greater continuity properties. Bézier curves and surfaces are commonly used in computer-aided design [1,2], image processing [3,4], and finite element modeling (e.g. [5–7]). Many other authors have studied Bézier curves and surfaces in Euclidean space (e.g. [8–12]).

In many problems in physics and mechanics, important functionals are often defined on surfaces or on other multidimensional objects. For example, the space of surfaces with a prescribed border is often studied. An important property studied on surfaces is that of their areas via the various area functionals. Another property relates to the Dirichlet functional, which is defined by the Lagrange functional based on the Laplacian operator. The relationship between both of these functionals is very similar to the relationship between length and energy functionals. The study of surfaces minimizing the area functional with prescribed border, called the Plateau problem, is to date a main topic in Euclidian differential geometry. Such kinds of surface, characterized by the vanishing mean curvature, are called minimal surfaces.

Recently, there has been interest in the relevant research communities in studying Bézier surfaces in various spaces as well as the properties of such surfaces subject to functionals. For example, Monterde studied Bézier surfaces of minimal area in $\mathbb{R}^3$ [10,11]. Miao et al. have studied the variational problems of finding Bézier surfaces that minimize the bending energy functional with prescribed border for both triangular and rectangular cases [13]. Xu and Wang have studied the properties of harmonic-type Bézier surfaces over both rectangular and triangular domains [14]. Farin and Hansford proposed a mask derived from the discretization of the Laplacian operator for generating the control net of the resulting Bézier surface [15]. Similarly, Ugail and Monterde studied the generation of Bézier surfaces satisfying the standard Laplace equation as well as the biharmonic equation, which they refer to as harmonic and biharmonic Bézier surfaces, respectively [16]. Later, they

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presented a method for generating Bézier surfaces from the boundary information based on a general fourth-order elliptic partial differential equation, and they point out that both the harmonic and biharmonic Bézier surfaces are related to minimal surfaces, i.e., surfaces that minimize the area among all the surfaces with prescribed boundary data [12].

In a somewhat parallel fashion, Minkowski space is the basis for the study of the physical phenomena described by the theory of relativity which has great geometric and physical meaning. Much work to date, therefore, has been done on timelike and spacelike surfaces in $\mathbb{R}^3_1$ (the three-dimensional Minkowski space). For example, Treibergs has studied spacelike hypersurfaces with constant mean curvature in Minkowski space [17]. Aledo et al. obtain a Lelievvre-type representation for timelike surfaces with prescribed Gauss map [18]. Abdel-Baky and Abd-Ellah study both spacelike and timelike ruled W-surfaces in $\mathbb{R}^3_1$ which satisfies a nontrivial relation between elements of the set $\{K, K_{II}, H, H_{II}\}$, where (K, H) and (K_{II}, H_{II}) are the Gaussian and mean curvatures of the first and the second fundamental forms [19]. Brander et al. constructed the spacelike constant mean curvature surfaces in $\mathbb{R}^3_1 SU(2)$ with the non-compact real form $SU(1, 1)$ [20]. Lin studied the implications of curvature restrictions on timelike surfaces in $\mathbb{R}^3_1$ that are convex as surfaces in Euclidean 3-space $\mathbb{R}^3$ [21]. Kossowski has studied restrictions on zero mean curvature surfaces in $\mathbb{R}^3_1$ [22]. Georgiev, in his recent work, obtained sufficient conditions for Bézier surfaces to be spacelike [23].

Given the vast studies that have been undertaken both for Bézier curves and surfaces as well as surfaces in Minkowski space, our motivation for this work stems in trying to understand the relationship between the two, i.e. we are interested in understanding the properties of Bézier surfaces in Minkowski space. To do this we first establish the necessary and sufficient conditions for a Bézier surface to exist in Minkowski space. We also study the Plateau problem for Bézier surfaces in Minkowski space by using the Dirichlet functional formulated in the metric of this space.

The paper is organised as follows. In Section 1, we first introduce basic notation and recall the conditions of timelike and spacelike surfaces in $\mathbb{R}^3_1$ Minkowski space. Furthermore, we give the definitions and derivative formulas of Bézier curves and surfaces. In Section 2, we define the first fundamental coefficients E, F, and G in terms of coordinates of the control points of the Bézier surface, and then we give the conditions of the timelike case and the spacelike case for Bézier surfaces. In Section 3, we deal with the Plateau–Bézier problem by obtaining conditions on the control points to be extremal of the Dirichlet functional. In Section 4, we give some examples for these cases, and we compare the area functionals for the minimal Bézier surface in $\mathbb{R}^3$ and $\mathbb{R}^3_1$. Finally, in Section 5, we conclude the work presented in this paper.

1. Notation and preliminaries

Let $\mathbb{R}^3_1$ be the three-dimensional Minkowski space, that is, the three-dimensional real vector space $\mathbb{R}^3$ with the metric

$$\langle , \rangle = (dx_1)^2 + (dx_2)^2 - (dx_3)^2,$$

where (x_1, x_2, x_3) denotes the canonical coordinates in $\mathbb{R}^3$. An arbitrary vector $\overrightarrow{v} = (v_1, v_2, v_3)$ in $\mathbb{R}^3_1$ can have one of three Lorentzian casual characters, i.e. it can be spacelike if $\langle \overrightarrow{v}, \overrightarrow{v} \rangle > 0$ or $\overrightarrow{v} = 0$, timelike if $\langle \overrightarrow{v}, \overrightarrow{v} \rangle < 0$, and null (lightlike) if $\langle \overrightarrow{v}, \overrightarrow{v} \rangle = 0$ and $\overrightarrow{v} \neq 0$. For any vectors $\overrightarrow{x} = (x_1, x_2, x_3)$ and $\overrightarrow{y} = (y_1, y_2, y_3)$ in $\mathbb{R}^3_1$, the Lorentz vector product of $\overrightarrow{x}$ and $\overrightarrow{y}$ is defined by

$$\overrightarrow{x} \times \overrightarrow{y} = \begin{vmatrix} -e_1 & -e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \left(- \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix}, \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix}, \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \right).$$

Let M be a surface in $\mathbb{R}^3_1$. M is called a timelike surface if the induced metric on M is a Lorentzian metric on each tangent plane. This is equivalent to saying that the unit normal vector N is spacelike at each point of M. M is called a spacelike surface if the induced metric on M is a positively definite Riemannian metric on each tangent plane. This is equivalent to saying that the unit normal vector N is timelike at each point of M. If M is given with a parameterization,

$$\Phi : U \subset \mathbb{R}^2 \to \mathbb{R}^3_1$$
$$(u, v) \to \Phi(u, v) = (\Phi_1(u, v), \Phi_2(u, v), \Phi_3(u, v)),$$

then the unit normal vector field N on M is given by

$$N = \frac{\Phi_u \times \Phi_v}{\|\Phi_u \times \Phi_v\|},$$

where $\Phi_u = \partial\Phi/\partial u$, $\Phi_v = \partial\Phi/\partial v$, and $\times$ stands for the Lorentzian cross product of $\mathbb{R}^3_1$.

The metric $\langle , \rangle$ on each tangent plane of M is determined by the first fundamental form,

$$I = \langle d\Phi, d\Phi \rangle = Edu^2 + 2Fdudv + Gdv^2,$$

with the differentiable coefficients

$$E = \langle \Phi_u, \Phi_u \rangle, \qquad F = \langle \Phi_u, \Phi_v \rangle, \qquad G = \langle \Phi_v, \Phi_v \rangle. \tag{1}$$

Since M is timelike and spacelike, we have

$$\det I = EG - F^2 < 0 \quad \text{and} \quad \det I = EG - F^2 > 0,$$

respectively.

Definition 1. Given $n + 1$ control points $P_0, P_1, \ldots, P_n$, the Bézier curve of degree n is defined to be

$$P(t) = \sum_{i=0}^{n} P_i B_i^n(t),$$

where

$$B_i^n(t) = \begin{cases} \dfrac{n!}{(n-i)!i!}(1-t)^{n-i}t^i, & \text{if } 0 \le i \le n \\ 0 & \text{otherwise} \end{cases}$$

are called the Bernstein polynomials or Bernstein basis functions of degree n [24].

Definition 2. Let $B_i^n(u)$ and $B_j^m(v)$ be the Bernstein basis functions of degree n and m, respectively. A Bézier surface with the control points P_{ij} $(0 \le i \le n, 0 \le j \le m)$ is the parametric surface defined by

$$P(u, v) = \sum_{i=0}^{n} \sum_{j=0}^{m} P_{ij} B_i^n(u) B_j^m(v),$$

for $(u, v) \in [0, 1] \times [0, 1]$.

Theorem 1. *The first derivative of a Bézier curve of degree n is*

$$P'(t) = \sum_{i=0}^{n-1} P_i^{(1)} B_i^{n-1}(t),$$

where $P_i^{(1)} = n.(P_{i+1} - P_i)$.

Theorem 2. *The partial derivatives* $P_u(u, v)$ *and* $P_v(u, v)$ *of a Bézier surface are obtained from the derivative formula for Bézier curves. Then,*

$$P_u(u, v) = \sum_{j=0}^{m} \left[n. \sum_{i=0}^{n-1} (P_{i+1,j} - P_{ij}) B_i^{n-1}(u) \right] B_j^m(v)$$
$$= \sum_{i=0}^{n-1} \sum_{j=0}^{m} P_{ij}^{(1,0)} B_i^{n-1}(u) B_j^m(v), \tag{2}$$

where $P_{ij}^{(1,0)} = n.(P_{i+1,j} - P_{ij})$. *Likewise, letting* $P_{ij}^{(0,1)} = m.(P_{i,j+1} - P_{ij})$,

$$P_v(u, v) = \sum_{i=0}^{n} \sum_{j=0}^{m-1} P_{ij}^{(0,1)} B_i^n(u) B_j^{m-1}(v). \tag{3}$$

2. Timelike and spacelike Bézier surfaces in $\mathbb{R}_1^3$ Minkowski space

In this section, we shall determine the timelike case and the spacelike case for Bézier surfaces. We shall need a lemma first. Let us compute how a Bernstein polynomial can be written as a linear combination of Bernstein polynomials of higher degree.

Lemma 1. *Given a Bernstein polynomial* $B_i^{n-k}(t)$, *we have*

$$B_i^{n-k}(t) = \sum_{l=0}^{k} \frac{\binom{n-i-l}{k-l}\binom{i+l}{l}}{\binom{n}{k}} B_{i+l}^n(t)$$
$$= \binom{n-k}{i} \sum_{l=0}^{k} \frac{\binom{k}{l}}{\binom{n}{i+l}} B_{i+l}^n(t),$$

for any $n > 0$, $k \in \{0, 1, \ldots, n\}$ *and* $i \in \{0, 1, \ldots, n-k\}$ [16].

Theorem 3. *Let $R(u,v)$ and $S(u,v)$ be surfaces in $\mathbb{R}^3$ defined by*

$$R(u,v)=\sum_{i=0}^{n}\sum_{j=0}^{m}a_{ij}B_i^n(u)B_j^m(v)$$

and

$$S(u,v)=\sum_{i=0}^{n}\sum_{j=0}^{m}b_{ij}B_i^n(u)B_j^m(v),$$

where $\{a_{ij}\}_{i,j=0}^{n,m}$ and $\{b_{ij}\}_{i,j=0}^{n,m}$ in $\mathbb{R}$. Then the product of these surfaces is defined by

$$R(u,v)\times S(u,v)=\sum_{i=0}^{n+n}\sum_{j=0}^{m+m}\left(\sum_{k=0}^{i}\sum_{l=0}^{j}\frac{\binom{n}{k}\binom{n}{i-k}\binom{m}{l}\binom{m}{j-l}a_{kl}b_{(i-k),(j-l)}}{\binom{2n}{i}\binom{2m}{j}}\right)B_i^{2n}(u)B_j^{2m}(v),$$

for $0\le k, i-k\le n$ and $0\le l, j-l\le m$.

We want to find the first fundamental form coefficients E, F, and G, which are defined in Eq. (1) for Bézier surfaces. First, we should calculate the partial derivatives $P_u(u,v)$ and $P_v(u,v)$ of a Bézier surface.

We shall rewrite Eqs. (2) and (3) again as a Bézier surface of degree n and m. In order to do this, we will need to apply the above lemma. Then, we have

$$\begin{aligned}B_i^{n-1}(u)&=\binom{n-1}{i}\sum_{l=0}^{1}\frac{\binom{1}{l}}{\binom{n}{i+l}}B_{i+l}^n(u)\\&=\frac{(n-1)!}{(n-1-i)!i!}\left(\frac{1}{\binom{n}{i}}B_i^n(u)+\frac{1}{\binom{n}{i+1}}B_{i+1}^n(u)\right)\\&=\frac{(n-1)!}{(n-1-i)!i!}\left[\frac{(n-i)!i!}{n.(n-1)!}B_i^n(u)+\frac{(n-i-1)!(i+1)!}{n.(n-1)!}B_{i+1}^n(u)\right]\\&=\frac{1}{n}.[(n-i).B_i^n(u)+(i+1).B_{i+1}^n(u)]\end{aligned}\tag{4}$$

and

$$B_j^{m-1}(v)=\frac{1}{m}.[(m-j).B_j^m(v)+(j+1).B_{j+1}^m(v)],\tag{5}$$

for $k\in\{0,1\}$ and $i,j\in\{0\}$.

Therefore, the $P_u(u,v)$ derivative given in Eq. (2) becomes

$$P_u(u,v)=\sum_{i=0}^{n}\sum_{j=0}^{m}[(n-i)P_{ij}^{(1,0)}+i.P_{i-1,j}^{(1,0)}]B_i^n(u)B_j^m(v).\tag{6}$$

In a similar fashion, we have

$$P_v(u,v)=\sum_{i=0}^{n}\sum_{j=0}^{m}[(m-j)P_{ij}^{(0,1)}+j.P_{i,j-1}^{(0,1)}]B_i^n(u)B_j^m(v).\tag{7}$$

These expressions can be seen as the Bézier surfaces associated with nets of control points $\{(n-i)P_{ij}^{(1,0)}+i.P_{i-1,j}^{(1,0)}\}_{i,j=0}^{n,m}$ and $\{(m-j)P_{ij}^{(0,1)}+j.P_{i,j-1}^{(0,1)}\}_{i,j=0}^{n,m}$, respectively.

Since we study surfaces in $\mathbb{R}_1^3$ Minkowski space, the control nets $\{(n-i)P_{ij}^{(1,0)}+i.P_{i-1,j}^{(1,0)}\}_{i,j=0}^{n,m}$ and $\{(m-j)P_{ij}^{(0,1)}+j.P_{i,j-1}^{(0,1)}\}_{i,j=0}^{n,m}$ are in $\mathbb{R}_1^3$. We can write Eqs. (6) and (7) as

$$\begin{aligned}P_u(u,v)=&\left(\sum_{i=0}^{n}\sum_{j=0}^{m}[(n-i)x_{ij}^{(1,0)}+i.x_{i-1,j}^{(1,0)}]B_i^n(u)B_j^m(v),\right.\\&\left.\times\sum_{i=0}^{n}\sum_{j=0}^{m}[(n-i)y_{ij}^{(1,0)}+i.y_{i-1,j}^{(1,0)}]B_i^n(u)B_j^m(v),\sum_{i=0}^{n}\sum_{j=0}^{m}[(n-i)z_{ij}^{(1,0)}+i.z_{i-1,j}^{(1,0)}]B_i^n(u)B_j^m(v)\right)\end{aligned}$$

and

$$P_v(u,v) = \left(\sum_{i=0}^{n}\sum_{j=0}^{m}[(m-j)x_{ij}^{(0,1)} + j.x_{i,j-1}^{(0,1)}]B_i^n(u)B_j^m(v),\right.$$
$$\left.\times \sum_{i=0}^{n}\sum_{j=0}^{m}[(m-j)y_{ij}^{(0,1)} + j.y_{i,j-1}^{(0,1)}]B_i^n(u)B_j^m(v), \sum_{i=0}^{n}\sum_{j=0}^{m}[(m-j)z_{ij}^{(0,1)} + j.z_{i,j-1}^{(0,1)}]B_i^n(u)B_j^m(v)\right),$$

where (x, y, z) denotes the canonical coordinates in $\mathbb{R}^3$.

Now we are able to calculate the first fundamental coefficients,

$$E = \langle P_u, P_u\rangle = \left(\sum_{i=0}^{n}\sum_{j=0}^{m}[(n-i)x_{ij}^{(1,0)} + i.x_{i-1,j}^{(1,0)}]B_i^n(u)B_j^m(v)\right)^2$$
$$+\left(\sum_{i=0}^{n}\sum_{j=0}^{m}[(n-i)y_{ij}^{(1,0)} + i.y_{i-1,j}^{(1,0)}]B_i^n(u)B_j^m(v)\right)^2$$
$$-\left(\sum_{i=0}^{n}\sum_{j=0}^{m}[(n-i)z_{ij}^{(1,0)} + i.z_{i-1,j}^{(1,0)}]B_i^n(u)B_j^m(v)\right)^2. \tag{8}$$

We shall now apply Theorem 3 to Eq. (8). Then, we have

$$E = \langle P_u, P_u\rangle = \sum_{i=0}^{2n}\sum_{j=0}^{2m}\sum_{k=0}^{i}\sum_{l=0}^{j}$$
$$\times\left(\frac{\binom{n}{k}\binom{n}{i-k}\binom{m}{l}\binom{m}{j-l}[(n-k).x_{kl}^{(1,0)} + k.x_{k-1,l}^{(1,0)}].[(n-i-k).x_{(i-k),(j-l)}^{(1,0)} + (i-k).x_{(i-k-1),(j-l)}^{(1,0)}]}{\binom{2n}{i}\binom{2m}{j}}\right)B_i^{2n}(u)B_j^{2m}(v)$$
$$+\left(\frac{\binom{n}{k}\binom{n}{i-k}\binom{m}{l}\binom{m}{j-l}[(n-k).y_{kl}^{(1,0)} + k.y_{k-1,l}^{(1,0)}].[(n-i-k).y_{(i-k),(j-l)}^{(1,0)} + (i-k).y_{(i-k-1),(j-l)}^{(1,0)}]}{\binom{2n}{i}\binom{2m}{j}}\right)B_i^{2n}(u)B_j^{2m}(v)$$
$$-\left(\frac{\binom{n}{k}\binom{n}{i-k}\binom{m}{l}\binom{m}{j-l}[(n-k).z_{kl}^{(1,0)} + k.z_{k-1,l}^{(1,0)}].[(n-i-k).z_{(i-k),(j-l)}^{(1,0)} + (i-k).z_{(i-k-1),(j-l)}^{(1,0)}]}{\binom{2n}{i}\binom{2m}{j}}\right)B_i^{2n}(u)B_j^{2m}(v).$$

We have

$$E = \langle P_u, P_u\rangle = \sum_{i=0}^{2n}\sum_{j=0}^{2m}\left[\sum_{k=0}^{i}\sum_{l=0}^{j}\frac{\binom{n}{k}\binom{n}{i-k}\binom{m}{l}\binom{m}{j-l}}{\binom{2n}{i}\binom{2m}{j}}\right.$$
$$\times([(n-k).x_{kl}^{(1,0)} + k.x_{k-1,l}^{(1,0)}].[(n-i+k).x_{(i-k),(j-l)}^{(1,0)} + (i-k).x_{(i-k-1),(j-l)}^{(1,0)}]$$
$$+[(n-k).y_{kl}^{(1,0)} + k.y_{k-1,l}^{(1,0)}].[(n-i+k).y_{(i-k),(j-l)}^{(1,0)} + (i-k).y_{(i-k-1),(j-l)}^{(1,0)}]$$
$$\left.-\ [(n-k).z_{kl}^{(1,0)} + k.z_{k-1,l}^{(1,0)}].[(n-i+k).z_{(i-k),(j-l)}^{(1,0)} + (i-k).z_{(i-k-1),(j-l)}^{(1,0)}])\right]B_i^{2n}(u)B_j^{2m}(v), \tag{9}$$

for $0 \le k, i-k \le n$ and $0 \le l, j-l \le m$.

In a similar way, we have

$$G = \langle P_v, P_v\rangle = \sum_{i=0}^{2n}\sum_{j=0}^{2m}\left[\sum_{k=0}^{i}\sum_{l=0}^{j}\frac{\binom{n}{k}\binom{n}{i-k}\binom{m}{l}\binom{m}{j-l}}{\binom{2n}{i}\binom{2m}{j}}\right.$$
$$\times[(m-l).x_{kl}^{(0,1)} + l.x_{k,l-1}^{(0,1)}].[(m-j+l).x_{(i-k),(j-l)}^{(0,1)} + (j-l).x_{(i-k),(j-l-1)}^{(0,1)}]$$
$$+[(m-l).y_{kl}^{(0,1)} + l.y_{k,l-1}^{(0,1)}].[(m-j+l).y_{(i-k),(j-l)}^{(0,1)} + (j-l).y_{(i-k),(j-l-1)}^{(0,1)}]$$
$$\left.-\ ([(m-l).z_{kl}^{(0,1)} + l.z_{k,l-1}^{(0,1)}].[(m-j+l).z_{(i-k),(j-l)}^{(0,1)} + (j-l).z_{(i-k),(j-l-1)}^{(0,1)}])\right]B_i^{2n}(u)B_j^{2m}(v) \tag{10}$$

and

$$F = \langle P_u, P_v \rangle = \sum_{i=0}^{2n} \sum_{j=0}^{2m} \left[\sum_{k=0}^{i} \sum_{l=0}^{j} \frac{\binom{n}{k}\binom{n}{i-k}\binom{m}{l}\binom{m}{j-l}}{\binom{2n}{i}\binom{2m}{j}} \right.$$
$$\times ([(n-k).x_{kl}^{(1,0)} + k.x_{k-1,l}^{(1,0)}].[(m-j+l).x_{(i-k),(j-l)}^{(0,1)} + (j-l).x_{(i-k),(j-l-1)}^{(0,1)}]$$
$$+[(n-k).y_{kl}^{(1,0)} + k.y_{k-1,l}^{(1,0)}].[(m-j+l).y_{(i-k),(j-l)}^{(0,1)} + (j-l).y_{(i-k),(j-l-1)}^{(0,1)}]$$
$$\left. -\ [(n-k).z_{kl}^{(1,0)} + k.z_{k-1,l}^{(1,0)}].[(m-j+l).z_{(i-k),(j-l)}^{(0,1)} + (j-l).z_{(i-k),(j-l-1)}^{(0,1)}]) \right] B_i^{2n}(u)B_j^{2m}(v), \quad (11)$$

for $0 \leq k, i-k \leq n$ and $0 \leq l, j-l \leq m$.

Theorem 4. *Let $P(u,v)$ be a Bézier surface in $\mathbb{R}_1^3$ three-dimensional Minkowski space. For $(u,v) \in [0,1]\times[0,1]$, $P(u,v)$ is called a timelike surface if*

$$F^2 - EG > 0,$$

where E, F, and G are defined in Equations (9), (10) and (11), respectively.

Theorem 5. *Let $P(u,v)$ be a Bézier surface in $\mathbb{R}_1^3$ three-dimensional Minkowski space. For $(u,v) \in [0,1]\times[0,1]$, $P(u,v)$ is called a spacelike surface if*

$$F^2 - EG < 0,$$

where E, F, and G are defined in Equations (9), (10) and (11), respectively.

3. Plateau–Bézier problem in $\mathbb{R}_1^3$ Minkowski space

In this section, we deal with the Plateau problem for Bézier surfaces in three-dimensional Minkowski space. Given the boundary points of a control net, we aim to find a Bézier surface (or the inner control points), such that it is extremal for the area function from among all the Bézier surfaces with the same boundary control points, in the space $\mathbb{R}_1^3$. This problem has been studied in the Euclidean space $\mathbb{R}^3$ by Monterde [10,11], obtaining the solution as the extremal of the Dirichlet functional.

Given a control net in $\mathbb{R}_1^3$, $\mathcal{P} = \{P_{ij}\}_{i,j=0}^{n,m}$, the associated Bézier surface, $\overrightarrow{X} : [0,1]\times[0,1] \to \mathbb{R}_1^3$, is

$$\overrightarrow{X}(u,v) = \sum_{i=0}^{n}\sum_{j=0}^{m} P_{ij}B_i^n(u)B_j^m(v).$$

The area of the Bézier surface in $\mathbb{R}_1^3$ is

$$\mathcal{A}(\mathcal{P}) = \int_R \|\overrightarrow{X}_u \times \overrightarrow{X}_v\|\, du\, dv = \int_R \sqrt{|\langle \overrightarrow{X}_u \times \overrightarrow{X}_v, \overrightarrow{X}_u \times \overrightarrow{X}_v \rangle|}\, du\, dv$$
$$= \int_R \sqrt{|EG - F^2|}\, du\, dv,$$

where $R = [0,1]\times[0,1]$ and E, G, and F are the coefficients of the first fundamental form of $\overrightarrow{X}$.

Due to the nonlinearity of the area function we encounter here, it can be very hard to compute its extremes. Instead, we will compute the extremes of a linear function greater than the area function.

3.1. Plateau–Bézier problem for spacelike surfaces

By definition, spacelike surfaces in $\mathbb{R}_1^3$ verify $F^2 - EG < 0$; then $|EG - F^2| = EG - F^2 > 0$ and $EG \geq EG - F^2 > 0$. Also, it is easy to see that $EG \leq \frac{(E+G)^2}{4}$. Therefore, we have

$$(EG - F^2)^{\frac{1}{2}} \leq (EG)^{\frac{1}{2}} \leq \frac{|E+G|}{2}.$$

Then, for any control net $\mathcal{P}$ defining a spacelike surface, we obtain

$$\mathcal{A}(\mathcal{P}) = \int_R \sqrt{EG - F^2}\, du\, dv \leq \frac{1}{2}\int_R |E+G|\, du\, dv.$$

The Dirichlet functional is defined by

$$\mathcal{D}(\mathcal{P}) = \frac{1}{2}\int_R (\|\overrightarrow{X}_u\|^2 + \|\overrightarrow{X}_v\|^2)\, du\, dv.$$

In $\mathbb{R}^3_1$ Minkowski space, this function can be written as

$$\mathcal{D}(\mathcal{P}) = \frac{1}{2}\int_R |\langle \overrightarrow{X}_u, \overrightarrow{X}_u\rangle| + |\langle \overrightarrow{X}_v, \overrightarrow{X}_v\rangle|\, du\, dv = \frac{1}{2}\int_R |E| + |G|\, du\, dv.$$

Spacelike surfaces verify $EG > 0$; then $|E+G| = |E| + |G|$. So

$$\mathcal{A}(\mathcal{P}) \le \mathcal{D}(\mathcal{P}).$$

Remark. Since $EG > 0$, it follows that the function $|E+G| \neq 0$, and, as a consequence, $|E+G|$ is differentiable for all $\mathcal{P}$ defining a spacelike Bézier surface.

In the next result, we obtain the conditions over the points of the control net to be extremal of the Dirichlet function, i.e. we obtain linear relations on the points.

Theorem 6. *A control net, $\mathcal{P}^{n,m}_{i,j=0}$, of a spacelike surface in $\mathbb{R}^3_1$ Minkowski space is an extremal of the Dirichlet function with fixed border if and only if*

$$0 = \frac{n^2}{2(2n-1)(2m+1)}\binom{n-1}{i}\binom{m}{j}\sum_{k,l=0}^{n-1,m} A^k_{n,i}\frac{\binom{m}{l}}{\binom{2m}{j+l}}P^{(1,0)}_{kl} + \frac{m^2}{2(2m-1)(2n+1)}\binom{n}{i}\binom{m-1}{j}\sum_{k,l=0}^{n,m-1}\frac{\binom{n}{k}}{\binom{2n}{i+k}}A^l_{mj}P^{(0,1)}_{kl},$$

for $i \in \{1,\ldots,n-1\}$ and $j \in \{1,\ldots,m-1\}$, where

$$A^k_{n,i} = \frac{ni - nk - i}{(n-i)(2n-1-i-k)}\frac{\binom{n-1}{k}}{\binom{2n-2}{i+k-1}}$$

and $P^{(1,0)}_{kl} = n(P_{k+1,l} - P_{kl})$, $P^{(0,1)}_{kl} = m(P_{k,l+1} - P_{kl})$.

Proof. Using the same notation, in $\mathbb{R}^3_1$ space we can consider the following two cases, depending on the sign of $|E+G|$:

$$\frac{\partial\mathcal{D}(\mathcal{P})}{\partial x^a_{ij}} = \int_R \left(\left\langle \frac{\partial \overrightarrow{X}_u}{\partial x^a_{ij}}, \overrightarrow{X}_u\right\rangle + \left\langle \frac{\partial \overrightarrow{X}_v}{\partial x^a_{ij}}, \overrightarrow{X}_v\right\rangle\right) du\, dv$$

or

$$\frac{\partial\mathcal{D}(\mathcal{P})}{\partial x^a_{ij}} = -\int_R \left(\left\langle \frac{\partial \overrightarrow{X}_u}{\partial x^a_{ij}}, \overrightarrow{X}_u\right\rangle + \left\langle \frac{\partial \overrightarrow{X}_v}{\partial x^a_{ij}}, \overrightarrow{X}_v\right\rangle\right) du\, dv.$$

For these two cases, we obtain the same result when we impose the condition $\frac{\partial\mathcal{D}(\mathcal{P})}{\partial x^a_{ij}} = 0$. □

Corollary 2. *A square control net, $\mathcal{P}^{n,n}_{i,j=0}$, of a spacelike surface in $\mathbb{R}^3_1$ Minkowski space is an extremal of the Dirichlet function with fixed border if and only if*

$$0 = \sum_{k,l=0}^{n-1,n}\frac{\binom{n}{l}}{\binom{2n}{j+l}}C^k_{ni}P^{(1,0)}_{kl} + \sum_{k,l=0}^{n,n-1}\frac{\binom{n}{k}}{\binom{2n}{i+k}}C^l_{nj}P^{(0,1)}_{kl},$$

for $i,j \in \{1,\ldots,n-1\}$ and $C^k_{ni} = \frac{(n-i)i-nk}{i+k}\frac{\binom{n-1}{k}}{\binom{2n-2}{i+k}}$.

For $n = m = 2$, we obtain the following condition for the control net:

$$P_{11} = \frac{1}{8}(3P_{00} - P_{01} + 3P_{02} - P_{10} - P_{12} + 3P_{20} - P_{21} + 3P_{22}).$$

For $n = m = 3$, we obtain the following conditions for the control net:

$$P_{11} = \frac{1}{78}(48P_{00} - 22P_{01} + 24P_{02} - 22P_{10} + 15P_{13} + 24P_{20} - 4P_{23} + 15P_{31} - 4P_{32} + 4P_{33}),$$
$$P_{12} = \frac{1}{78}(24P_{01} - 22P_{02} + 48P_{03} + 15P_{10} - 22P_{13} - 4P_{20} + 244P_{23} + 4P_{30} - 4P_{31} + 15P_{32}),$$
$$P_{21} = \frac{1}{78}(15P_{01} - 4P_{02} + 4P_{03} + 24P_{10} - 4P_{13} - 22P_{20} + 15P_{23} + 48P_{30} - 22P_{31} + 24P_{32}),$$
$$P_{22} = \frac{1}{78}(4P_{00} - 4P_{01} + 15P_{02} - 4P_{10} + 24P_{13} + 15P_{20} - 22P_{23} + 24P_{31} - 22P_{32} + 48P_{33}).$$

3.2. Plateau–Bézier problem for timelike surfaces

Timelike surfaces in $\mathbb{R}^3_1$ verify that $F^2 - EG > 0$; then, for any control net $\mathcal{P}$ defining a timelike surface, its area is

$$\mathcal{A}(\mathcal{P}) = \int_R \sqrt{F^2 - EG}\, du\, dv.$$

In order to find extremals for a linear function greater than the area function, we are going to distinguish the following two cases.

- Timelike surfaces verifying $|EG| < F^2$.
  In this case, we have $F^2 - EG \leq 2F^2$; then $\sqrt{F^2 - EG} \leq \sqrt{2}\ |F|$. Since the function $|F| \neq 0$ is a timelike surfaces verifying $|EG| < F^2$, it follows that it is differentiable. We consider the function

  $$\mathcal{F}(\mathcal{P}) = \sqrt{2} \int_R |F|\, du\, dv,$$

  defined on the set consisting of all the points of the control net $\mathcal{P}$ defining timelike surfaces with $|EG| < F^2$. So,

  $$\mathcal{A}(\mathcal{P}) \leq \mathcal{F}(\mathcal{P}).$$

- Timelike surfaces verifying $|EG| > F^2$.
  Now, we have $F^2 - EG \leq 2|EG| \leq (|E| + |G|)^2$; then $\sqrt{F^2 - EG} \leq |E| + |G|$. Since $|EG| > F^2$, if we suppose that $E = 0$ or $G = 0$, it follows $F = 0$. But this is impossible in timelike surfaces. Then, $|E| + |G|$ is differentiable. Furthermore, it is easy to see that $|EG| = -EG$. So, it is clear that $|E| + |G| = |E - G|$.

Let us recall the Dirichlet function $\mathcal{D}(\mathcal{P})$ defined above. In this case, for all $\mathcal{P}$ defining timelike surfaces with $|EG| > F^2$, we have that $\mathcal{D}(\mathcal{P}) = \int_R |E - G|\, du\, dv$, so we conclude that

$$\mathcal{A}(\mathcal{P}) \leq 2\ \mathcal{D}(\mathcal{P}).$$

Theorem 7. *A control net, $\mathcal{P}^{n,m}_{i,j=0}$, of a timelike surface in $\mathbb{R}^3_1$ Minkowski space verifying $|EG| < F^2$ is an extremal of the function $\mathcal{F}(\mathcal{P})$ with fixed border if and only if*

$$0 = \binom{n-1}{i}\binom{m}{j} \sum_{k,l=0}^{n,m-1} \frac{ni - nk}{(n-i)(i+k)} \frac{\binom{n}{k}}{\binom{2n-1}{i+k}} \frac{\binom{m-1}{l}}{\binom{2m-1}{j+l}} P^{(0,1)}_{kl}$$
$$+ \binom{n}{i}\binom{m-1}{j} \sum_{k,l=0}^{n-1,m} \frac{mj - ml}{(m-j)(j+l)} \frac{\binom{n-1}{k}}{\binom{2n-1}{i+k}} \frac{\binom{m}{l}}{\binom{2m-1}{j+l}} P^{(1,0)}_{kl},$$

for $i \in \{1, \ldots, n-1\}$ and $j \in \{1, \ldots, m-1\}$.

Proof. We consider

$$\frac{\partial \mathcal{F}(\mathcal{P})}{\partial x^a_{ij}} = \sqrt{2} \int_R \frac{\partial}{\partial x^a_{ij}} |\langle \overrightarrow{X}_u, \overrightarrow{X}_v \rangle|\, du\, dv = \sqrt{2} \int_R \left| \left\langle \frac{\partial \overrightarrow{X}_u}{\partial x^a_{ij}}, \overrightarrow{X}_v \right\rangle + \left\langle \frac{\partial \overrightarrow{X}_v}{\partial x^a_{ij}}, \overrightarrow{X}_u \right\rangle \right| du\, dv$$
$$= \sqrt{2} \int_R |n\ (B^{n-1}_{i-1}(u) - B^{n-1}_i(u))\ B^m_j(v)\ \langle e^a, \overrightarrow{X}_v \rangle$$
$$+ m\ (B^{m-1}_{j-1}(v) - B^{m-1}_j(v))\ B^n_i(u)\ \langle e^a, \overrightarrow{X}_u \rangle|\, du\, dv$$

$$= \sqrt{2}(\, B^{n-1}_{i-1}(u) - B^{n-1}_{i}(u))\, B^{m}_{j}(v) \left\langle e^a, m \sum_{k,l=0}^{n,m-1} B^n_k(u)\, B^{m-1}_l(v)\, P^{(0,1)}_{kl} \right\rangle$$
$$+ m\, (B^{m-1}_{j-1}(v) - B^{m-1}_{j}(v))\, B^n_i(u) \left\langle e^a, n \sum_{k,l=0}^{n-1,m} B^{n-1}_k(u)\, B^m_l(v)\, P^{(1,0)}_{kl} \right\rangle |\, du\, dv.$$

Since $\int_0^1 B_i^n(t)\, dt = \frac{1}{n-1}$, it follows that

$$\frac{\partial \mathcal{F}(\mathcal{P})}{\partial x^a_{ij}} = \sqrt{2}\, n\, m \left| \frac{1}{2n}\frac{1}{2m} \sum_{k,l=0}^{n,m-1} \frac{\binom{m}{j}\binom{m-1}{l}}{\binom{2m-1}{j+l}} \left(\frac{\binom{n-1}{i-1}\binom{n}{k}}{\binom{2n-1}{i+k-1}} - \frac{\binom{n-1}{i}\binom{n}{k}}{\binom{2n-1}{i+k}} \right) \langle e^a, P^{(0,1)}_{kl} \rangle \right.$$
$$\left. + \frac{1}{2n}\frac{1}{2m} \sum_{k,l=0}^{n-1,m} \frac{\binom{n}{i}\binom{n-1}{k}}{\binom{2n-1}{i+k}} \left(\frac{\binom{m-1}{j-1}\binom{m}{l}}{\binom{2m-1}{j+l-1}} - \frac{\binom{m-1}{j}\binom{m}{l}}{\binom{2m-1}{j+l}} \right) \langle e^a, P^{(1,0)}_{kl} \rangle \right|$$
$$= \left| \frac{\sqrt{2}}{4} \binom{m}{j}\binom{n-1}{i} \sum_{k,l=0}^{n,m-1} \frac{ni - nk}{(n-i)(i+k)} \frac{\binom{n}{k}}{\binom{2n-1}{i+k}} \frac{\binom{m-1}{l}}{\binom{2m-1}{j+l}} \langle e^a, P^{(0,1)}_{kl} \rangle \right.$$
$$\left. + \frac{\sqrt{2}}{4} \binom{n}{i}\binom{m-1}{j} \sum_{k,l=0}^{n-1,m} \frac{mj - ml}{(m-j)(j+l)} \frac{\binom{m}{l}}{\binom{2m-1}{j+l}} \frac{\binom{n-1}{k}}{\binom{2n-1}{i+k}} \langle e^a, P^{(1,0)}_{kl} \rangle \right|. \quad \square$$

Corollary 3. *A square control net, $\mathcal{P}^{n,n}_{i,j=0}$, of a timelike surface in $\mathbb{R}^3_1$ Minkowski space with $|EG| < F^2$ is an extremal of the function $\mathcal{F}(\mathcal{P})$ with fixed border if and only if*

$$0 = \sum_{k,l=0}^{n,n-1} \frac{\binom{n}{k}\binom{n-1}{l}}{\binom{2n-1}{i+k}\binom{2n-1}{j+l}} \frac{ni - nk}{i+k} P^{(0,1)}_{kl} + \sum_{k,l=0}^{n-1,n} \frac{\binom{n}{l}\binom{n-1}{k}}{\binom{2n-1}{j+l}\binom{2n-1}{i+k}} \frac{n(j-l)}{j+l} P^{(1,0)}_{kl},$$

for $i, j \in \{1, \ldots, n-1\}$.

For $n = m = 2$, we obtain the following condition on the control net:

$$0 = -P_{00} + P_{02} + P_{20} - P_{22}.$$

Theorem 8. *A control net, $\mathcal{P}^{n,m}_{i,j=0}$, of a timelike surface in $\mathbb{R}^3_1$ Minkowski space $|EG| > F^2$ is an extremal of the Dirichlet function with fixed border if and only if*

$$0 = \frac{n^2}{2(2n-1)(2m+1)} \binom{n-1}{i}\binom{m}{j} \sum_{k,l=0}^{n-1,m} A^k_{n,i} \frac{\binom{m}{l}}{\binom{2m}{j+l}} P^{(1,0)}_{kl}$$
$$- \frac{m^2}{2(2m-1)(2n+1)} \binom{n}{i}\binom{m-1}{j} \sum_{k,l=0}^{n,m-1} \frac{\binom{n}{k}}{\binom{2n}{i+k}} A^l_{mj} P^{(0,1)}_{kl},$$

for $i \in \{1, \ldots, n-1\}$ and $j \in \{1, \ldots, m-1\}$, where

$$A^k_{n,i} = \frac{ni - nk - i}{(n-i)(2n-1-i-k)} \frac{\binom{n-1}{k}}{\binom{2n-2}{i+k-1}}.$$

Proof. Let us recall that timelike surfaces with $|EG| > F^2$ verify $\mathcal{A}(\mathcal{P}) \le 2\mathcal{D}(\mathcal{P}) = \int_R |E - G|\, du\, dv$. This proof is similar to the proof of Theorem 6. $\square$

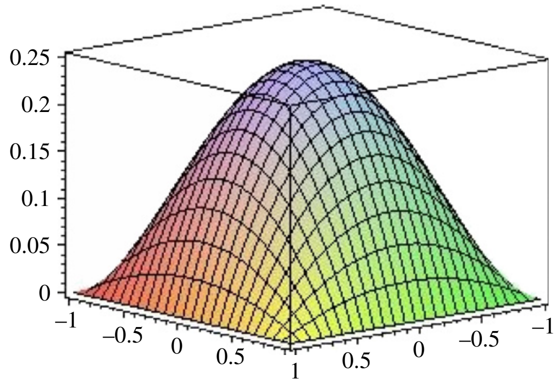

Fig. 1. A spacelike Bézier surface.

Corollary 4. *A square control net, $\mathcal{P}_{i,j=0}^{n,n}$, of a timelike surface in $\mathbb{R}_1^3$ Minkowski space with $|EG| > F^2$ is an extremal of the Dirichlet function with fixed border if and only if*

$$0 = \sum_{k,l=0}^{n-1,n} \frac{\binom{n}{l}}{\binom{2n}{j+l}} C_{ni}^k P_{kl}^{(1,0)} - \sum_{k,l=0}^{n,n-1} \frac{\binom{n}{k}}{\binom{2n}{i+k}} C_{nj}^l P_{kl}^{(0,1)},$$

for $i, j \in \{1, \ldots, n-1\}$ and $C_{ni}^k = \frac{(n-i)i-nk}{i+k} \frac{\binom{n-1}{k}}{\binom{2n-2}{i+k}}$.

For $n = m = 2$, we obtain the following condition on the control net:

$$0 = -P_{01} + P_{10} + P_{12} - P_{21}.$$

4. Examples

In this section, we discuss several examples of Bézier surfaces in $\mathbb{R}_1^3$ that are spacelike, timelike, and mixed type.

4.1. Example 1

Let a Bézier surface M be given with a control net $\mathcal{P} = \{P_{ij}\}_{i,j=0}^{2,2}$.

$$P_{00} = (-1, -1, 0) \quad P_{01} = (-1, 0, 0) \quad P_{02} = (-1, 1, 0),$$
$$P_{10} = (0, -1, 0) \quad P_{11} = (0, 0, 1) \quad P_{12} = (0, 1, 0),$$
$$P_{20} = (1, -1, 0) \quad P_{21} = (1, 0, 0) \quad P_{22} = (1, 1, 0).$$

We compute the first fundamental coefficients E, F, and G, which are given respectively in Equations (9), (10) and (11), in terms of coordinates of control points. For $(u, v) \in [0, 1] \times [0, 1]$, $F^2 - EG < 0$. Therefore, this surface, shown in Fig. 1, is a spacelike surface. If we calculate the Dirichlet function associated to this surface in $\mathbb{R}_1^3$, it is

$$\mathcal{D}(\mathcal{P}) = 3.822,$$

so, in three-dimensional Minkowski space, we have

$$\mathcal{A}(\mathcal{P}) \leq 3.822.$$

However, if we consider the spacelike Bézier surface M' with control net, $\mathcal{P}'$, defined by the same boundary control points as M but with a different inner control point at $P_{11} = (0, 0, 0)$, we obtain a flat surface. Indeed, this is the square projection of M on the plane $z = 0$. Its first fundamental coefficients are $E = G = 4$, $F = 0$ by Equations (9), (10) and (11). Then, $\mathcal{A}(\mathcal{P}') = 4$.

We conclude that in $\mathbb{R}_1^3$ the area of the projection M' of the spacelike surface M is greater than the area of M. Moreover, by Corollary 2, M' is extremal for the Dirichlet function over spacelike Bézier surfaces with the same boundary points as M and inner points $(0, 0, z)$ in three-dimensional Minkowski space. In fact, it is a maximum. However, it is well known that, in three-dimensional Euclidean space, M' is extremal for the area and the Dirichlet functions and that it is a minimum (see Fig. 2).

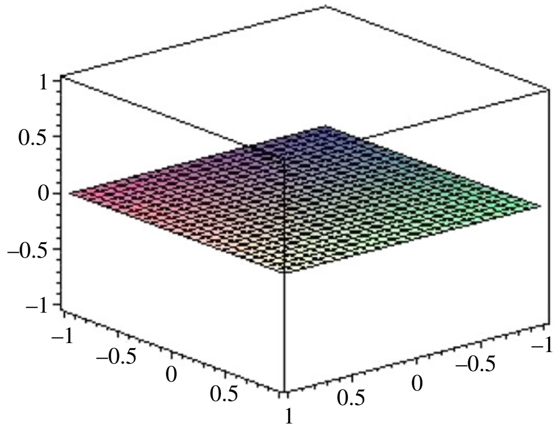

Fig. 2. The spacelike Bézier surface in Fig. 1 with different inner control point $P_{11} = (0, 0, 0)$.

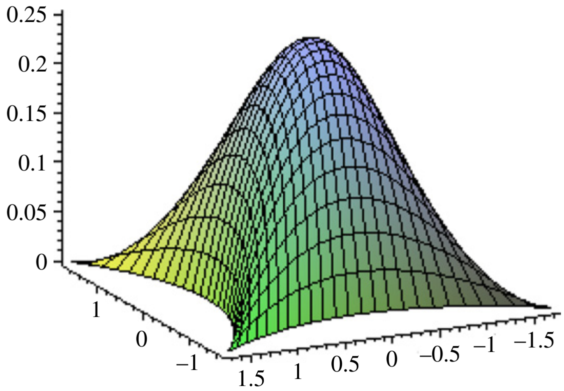

Fig. 3. A spacelike Bézier surface.

4.2. Example 2

Let a Bézier surface be given with a control net $\mathcal{P} = \{P_{ij}\}_{i,j=0}^{2,2}$.

$$P_{00} = \left(-1 - \frac{\sqrt{2}}{2}, -1 - \frac{\sqrt{2}}{2}, 0\right) \qquad P_{01} = \left(-\frac{1}{2}, 0, 0\right) \qquad P_{02} = \left(-1 - \frac{\sqrt{2}}{2}, 1 + \frac{\sqrt{2}}{2}, 0\right),$$

$$P_{10} = \left(0, -\frac{1}{2}, 0\right) \qquad P_{11} = (0, 0, 1) \qquad P_{12} = \left(0, \frac{1}{2}, 0\right),$$

$$P_{20} = \left(1 + \frac{\sqrt{2}}{2}, -1 - \frac{\sqrt{2}}{2}, 0\right) \qquad P_{21} = \left(\frac{1}{2}, 0, 0\right) \qquad P_{22} = \left(1 + \frac{\sqrt{2}}{2}, 1 + \frac{\sqrt{2}}{2}, 0\right).$$

We compute the first fundamental coefficients E, F, and G, which are given respectively in Equations (9), (10) and (11), in terms of coordinates of control points. For $(u, v) \in [0, 1] \times [0, 1]$, $F^2 - EG < 0$. Therefore, this surface, shown in Fig. 3, is a spacelike surface.

If we calculate the Dirichlet function associated to this surface in $\mathbb{R}_1^3$, it is

$$\mathcal{D}(\mathcal{P}) = 7.4087,$$

so, in three-dimensional Minkowski space, we have

$$\mathcal{A}(\mathcal{P}) \leq 7.4087.$$

However, by Corollary 2, the extremal for the Dirichlet function over spacelike Bézier surfaces with the same boundary points is obtained for the following surface:

The value of the Dirichlet function in this surface is

$$\mathcal{D}(\mathcal{P}') = 7.586487.$$

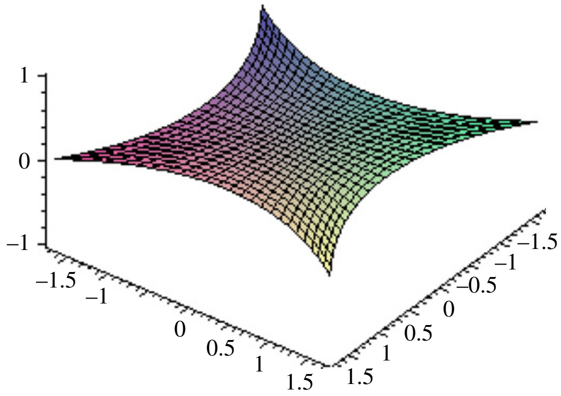

Fig. 4. A spacelike Bézier surface.

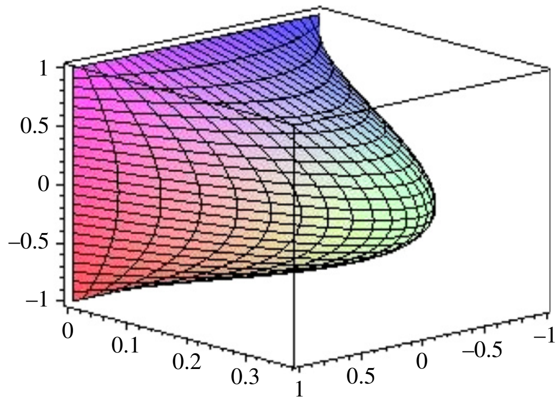

Fig. 5. A timelike Bézier surface.

4.3. Example 3

Let a Bézier surface be given with a control net $\mathcal{P} = \{P_{ij}\}_{i,j=0}^{2,2}$.

$$P_{00} = (1, 0, 1) \quad P_{01} = (0, 0, 1) \quad P_{02} = (-1, 0, 1)$$
$$P_{10} = (1, 0, 0) \quad P_{11} = (0, 1, 0) \quad P_{12} = (-1, 0, 0)$$
$$P_{20} = (1, 0, -1) \quad P_{21} = (0, 0, -1) \quad P_{22} = (-1, 0, -1).$$

We compute the first fundamental coefficients E, F, and G, which are given respectively in Equations (9), (10) and (11), in terms of coordinates of control points. For $(u, v) \in [0, 1] \times [0, 1]$, $F^2 - EG > 0$. Therefore, this surface, shown in Fig. 4, is a timelike surface. Furthermore, it verifies $F^2 < |EG|$.

Let us calculate the Dirichlet function:

$$\mathcal{D}(\mathcal{P}) = \frac{1}{2} \int_R |E - G| \, du \, dv = 4.$$

Therefore,

$$\mathcal{A}(\mathcal{P}) \leq 2 \, \mathcal{D}(\mathcal{P}) = 8.$$

By Corollary 3, we have that this surface is extremal for the Dirichlet function on timelike Bézier surfaces verifying $|EG| > F^2$ with the same boundary points. In fact, if we consider the Bézier surface with the same boundary control net but with the inner point $P_{11} = (0, \frac{3}{2}, 0)$ or $P_{11} = (0, 2, 0)$, we obtain, in both cases, a timelike surface verifying $|EG| > F^2$, and the value of the associated Dirichlet functions for both surfaces is 4. So, in this case, the Dirichlet function in $\mathbb{R}^3_1$ does not have a unique extremal.

Here, one would wish to find an example of a timelike minimal surface verifying $|EG| > F^2$, though this is not possible. This is because in this case the function which is greater than the area function is not the Dirichlet function. So, the extremals for the Dirichlet function, in this case, do not provide solutions to the associated Plateau problem.

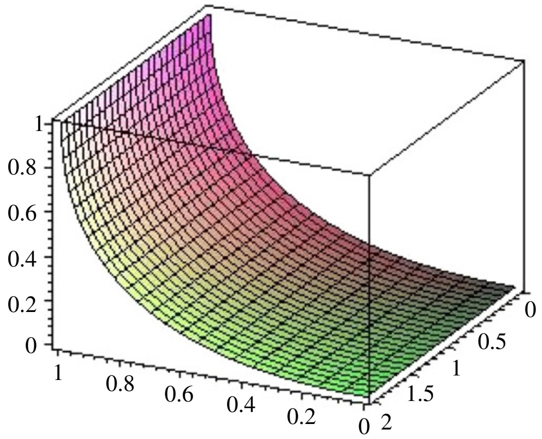

Fig. 6. A mixed-type Bézier surface.

4.4. Example 4

Let a Bézier surface be given with a control net $\{P_{ij}\}_{i,j=0}^{2,2}$.

$$\begin{aligned} P_{00} &= (0, 0, 0) & P_{01} &= (0, 1, 0) & P_{02} &= (0, 2, 0), \\ P_{10} &= (1, 0, 0) & P_{11} &= (1, 1, 0) & P_{12} &= (1, 2, 0), \\ P_{20} &= (1, 0, 1) & P_{21} &= (1, 1, 1) & P_{22} &= (1, 2, 1). \end{aligned}$$

We compute the first fundamental coefficients E, F, and G, which are given respectively in Equations (9), (10) and (11), in terms of coordinates of control points. For $u \in [0, 1]$, $v \in (\frac{1}{2}, 1]$, $F^2 - EG < 0$. So, this part of the surface is spacelike. For $u \in [0, 1]$, $v \in [0, \frac{1}{2})$, $F^2 - EG > 0$. So, this part of surface, shown in Fig. 5, is of mixed type. Note that, in this example, $v = \frac{1}{2}$, $u \in [0, 1]$ corresponds to a null curve (see Fig. 6).

5. Conclusion

In this paper, we have studied Bézier surfaces in $\mathbb{R}_1^3$ three-dimensional Minkowski space. This work is mainly geared to show that timelike and spacelike cases for Bézier surfaces by the corresponding control net of the surface. For this purpose, we defined the first fundamental coefficients E, F, and G in terms of coordinates of control points of the surfaces, and then we gave the conditions for the timelike case and the spacelike case.

Furthermore, we have studied the Plateau problem in timelike and spacelike Bézier surfaces by using the extremal of the Dirichlet functional in $\mathbb{R}_1^3$ Minkowski space. We have illustrated this study with examples through which we have compared Plateau–Bézier problems in $\mathbb{R}^3$ and $\mathbb{R}_1^3$.

In our future work under this theme, we propose to study the conditions on the control points of a Bézier surface to be unique extremal and maximum or minimum of the Dirichlet function. Also, we are interested in the study of the extremals of the Lagrange functional generating the harmonic and biharmonic Laplacian functions on Bézier surfaces in $\mathbb{R}_1^3$ Minkowski space. Furthermore, studies towards classification of timelike, spacelike and mixed-type cases will be interesting topics for future research.

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