



Explicit polynomial solutions of fourth order linear elliptic Partial Differential Equations for boundary based smooth surface generation [☆]

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ABSTRACT

We present an explicit polynomial solution method for surface generation. In this case the surface in question is characterized by some boundary configuration whereby the resulting surface conforms to a fourth order linear elliptic Partial Differential Equation, the Euler–Lagrange equation of a quadratic functional defined by a norm. In particular, the paper deals with surfaces generated as explicit Bézier polynomial solutions for the chosen Partial Differential Equation. To present the explicit solution methodologies adopted here we divide the Partial Differential Equations into two groups namely the orthogonal and the non-orthogonal cases. In order to demonstrate our methodology we discuss a series of examples which utilize the explicit solutions to generate smooth surfaces that interpolate a given boundary configuration. We compare the speed of our explicit solution scheme with the solution arising from directly solving the associated linear system.

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1. Introduction

This paper is concerned with an explicit polynomial solution method for surface generation which is characterized by some boundary whereby the resulting surface conforms to a linear elliptic Partial Differential Equation (PDE). In particular, we deal with surfaces generated as explicit Bézier polynomial solutions for a large class of fourth order linear PDEs.

There are many areas of science and engineering whereby physical phenomenon are modeled using fourth order linear elliptic PDEs. Examples include material sciences (e.g. stress/strain in physical structures, see Love, 1999; Gregory, 1992; Reissner, 1963), biology (e.g. fluid dynamics inside lungs, see Halpern et al., 1998), image processing (e.g. enhancement of noisy images, see You and Kaveh, 2000) and computer aided geometric design (e.g. surface design, see Bloor and Wilson, 1990; Du and Qin, 2004; Zhang and You, 2004; Monterde and Ugail, 2004). Popular methods for the explicit solution of the related PDEs involve the use of eigenfunction expansions, integral transforms, Green's functions and power series solutions.

The problem we are addressing here can be described as follows. Given four boundary curves find a parametric surface patch \vec{x} such that $\vec{x}: [u, v] \rightarrow \mathbf{R}^3$ whereby the surface patch \vec{x} smoothly interpolates the four curves. We assume the four boundary curves are defined as $\vec{x}(u, 0)$, $\vec{x}(u, 1)$, $\vec{x}(v, 0)$ and $\vec{x}(v, 1)$ where the domain of \vec{x} is the unit square $0 \leq u, v \leq 1$.

The above problem probably is as old as the field of computer aided geometric design. Thus, given the importance of the problem, not surprisingly, many solutions to this problem have been proposed. One of the earliest solutions to this problem is the Gordon surface (Gordon, 1969) which is a surface resulting from a linear interpolation between boundary curves. Similarly, Coons surfaces (Barnhill, 1982) are linear interpolations based on given boundary curves or Bicubically blended

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Coons patches that are defined by both boundary curves and cross-boundary derivatives (Farin, 2002). Various forms of Coons surfaces, both in continuous and discrete domain, have been proposed (Barnhill, 1982; Farin and Hansford, 1999).

When one is concerned with generating *smooth* surfaces conforming to a given boundary configuration, it is also common to pose the problem within a variational setting (Moreton and Séquin, 1992). Various basic functionals can be utilized for this purpose. For example, $\int_{[0,1]^2} \|\bar{\mathbf{x}}_u\|^2 + \|\bar{\mathbf{x}}_v\|^2 du dv$, corresponds to the Harmonic functional, or Dirichlet functional. Another typical example is $\int_{[0,1]^2} \|\bar{\mathbf{x}}_{uv}\|^2 du dv$, corresponding to the Coons functional which minimizes the surfaces twist (Nielson et al., 1978). Yet another example is, $\int_{[0,1]^2} \|\bar{\mathbf{x}}_{uu}\|^2 + 2\|\bar{\mathbf{x}}_{uv}\|^2 + \|\bar{\mathbf{x}}_{vv}\|^2 du dv$, which corresponds to the Biharmonic functional. Apart from the above common functionals, other higher order functionals, or functionals with added terms or with modifying parameters also have been utilized. A notable piece of work carried out in this area based on the Biharmonic functional which is widely known in the field of geometric modelling as the PDE method by Bloor and Wilson (1990). The PDE method solves a boundary-value problem subject to boundary conditions which can be taken to be four curves (Du and Qin, 2004; Zhang and You, 2004; Ugail, 2004). Here we will solve those PDEs being the Euler–Lagrange equations of a kind of quadratic functionals defined by a norm.

Our previous work on the theme of boundary based smooth surface design involved the development of methods for generating Bézier surfaces verifying elliptic boundary value problems. In particular for boundary value problems associated with the Laplace equation as well as the Biharmonic equation which are referred to as Harmonic and Biharmonic Bézier surfaces respectively (Monterde and Ugail, 2004). The main point we note from this previous work is that both the Harmonic and Biharmonic Bézier surfaces are related to minimal surfaces, i.e. surfaces that minimize the area among all the surfaces with prescribed boundary data. In the Harmonic case two boundary conditions are required to construct the surface. Similarly in the Biharmonic case four boundary conditions are required to satisfy the fourth order elliptic PDE. It is also important to highlight that for the Biharmonic case, even though the chosen boundary-value problem is of fourth order the knowledge of the boundaries defining the edges of surface patch alone enables one to fully determine the entire surface. This polynomial solution method has been also generalized to any fourth order Biharmonic equation (Monterde and Ugail, 2006).

It is noteworthy that in a classical sense, for example for the solution involving the Biharmonic equation, the function values and its normal derivatives at the boundaries are often what is considered as boundary conditions (e.g. Meleshko, 1998). In our case, however, we are solving a mixed boundary-value problem since our solution domain here is restricted to polynomial functions.

The aim of this paper is to describe a methodology to explicitly compute the polynomial solutions, especially Bézier solutions, of the fourth order linear elliptic PDEs satisfying given four boundary conditions. Thus, our proposed technique for surface generation from the boundary has the following desirable features. Firstly we are able to generate a surface satisfying the linear fourth order elliptic PDEs, thus, the resulting surface is naturally smooth and fair. Secondly our proposed solution framework here is of analytic nature and therefore the resolution of the solution is fast. Moreover, the availability of an explicit expression for the resulting surface implies that the requirement for arbitrary level of surface refinement can be carried out. Finally since our solution formulation is of polynomial in nature the resulting surfaces have a high degree of compatibility with modern Computer Aided Design data exchange schemes (Rappoport, 2003).

It is noteworthy to point out that we are indeed looking for polynomial solutions of the Biharmonic equation (not for arbitrary functions). Recalling this, by prescribing the values along the boundary, but not for the tangent plane, a polynomial solution of the Biharmonic equation is fixed. For example, lets suppose that we have a boundary parametrized by polynomial functions. Among all the solutions of the Biharmonic equation with that boundary, one for each possible configuration to the normal derivatives along the boundary, there is one and only one polynomial function satisfying this condition. Let's consider the following simple argument to explain this further.

Let $f(u, v)$ be a polynomial function of degree $\leq n$. To prescribe the boundary $f(0, v)$ is equivalent to fix $n + 1$ coefficients of f . To prescribe the four boundaries, $f(0, v)$, $f(1, v)$, $f(u, 0)$ and $f(u, 1)$ is equivalent to fix $4n$ coefficients. The Biharmonic operator can be written using complex coordinates $z = u + iv$, $\bar{z} = u - iv$, and up to constant factors as $\frac{\partial^4}{\partial z^2 \partial \bar{z}^2}$. A monomial of the kind $z^k \bar{z}^\ell$ is Biharmonic if and only if $k \leq 1$ or $\ell \leq 1$. Therefore, the subspace of polynomial solutions of the Biharmonic equation of degree $\leq n$ is generated by $\{z^k\}_{k=0}^n \cup \{z^k \bar{z}\}_{k=0}^n \cup \{\bar{z}^k\}_{k=2}^n \cup \{z \bar{z}^k\}_{k=2}^n$. This makes a basis of $n + 1 + n + 1 + n - 1 + n - 1 = 4n$ vectors. Therefore, if we fix the boundary, we fix the $4n$ scalars which are necessary to determine the unique Biharmonic polynomial solution.

Here we also would like to note similar recent work on this topic e.g. Idrees Bhatti and Bracken (2007) where the authors obtain polynomial approximations to ordinary differential equations using the Bernstein polynomial basis. However, in this work, though the solution obtained is of polynomial form, there exist two differences. Firstly, we obtain explicit solutions to some linear 4th order PDEs and secondly, the boundary conditions are not the usual ones from the point of view of PDEs.

The paper is organized as follows. In the next section we describe the method of polynomial solutions for the fourth order linear elliptic PDEs using the Bézier polynomial formulation. In order to present the explicit solution formulation efficiently we categorize the PDEs into two groups namely the orthogonal and the non-orthogonal cases as discussed in the next section. In Section 3 we translate the problem from boundary control points to the polynomial expression of a Bézier surface. Thus, in Section 4 we discuss the explicit polynomial solution for the cases involving the non-orthogonal conditions. In Section 5 we discuss explicit solutions with orthogonal conditions for the chosen PDE. In Section 6 we come back to the Bézier form. In Section 7 we discuss a series of examples which utilize the explicit solutions to generate surfaces from a

given boundary configuration. In particular, we compare the speed of our explicit solution scheme with solution arising from directly solving the associated linear system. Finally we conclude the paper.

2. Formulation of the problem

The Biharmonic functional is nothing but the sum of the squares of the norms of the four entries of the matrix

$$M := \begin{pmatrix} \vec{x}_{uu} & \vec{x}_{uv} \\ \vec{x}_{vu} & \vec{x}_{vv} \end{pmatrix},$$

and it can be understood as a norm of the matrix M .

Indeed, the Frobenius norm, which we shall denote by the subindex Fr , sometimes known as the Euclidean norm, is a matrix norm of an $m \times n$ matrix $M = (m_{ij})_{i,j=1}^{m,n}$ defined as the square root of the sum of the absolute squares of its elements, i.e.,

$$\|M\|_{Fr} := \left(\sum_{i=1}^m \sum_{j=1}^n m_{ij}^2 \right)^{\frac{1}{2}}.$$

It is also equivalent to the square root of the matrix trace of $M.M^T$, where M^T is the transpose, i.e.,

$$\|M\|_{Fr} := \sqrt{tr(M.M^T)},$$

and where tr denotes the trace.

In order to obtain different matrix norms one can modify the Frobenius norm as follows. Given a nonsingular matrix G we define the G -norm of that matrix M as

$$\|M\|_G := \sqrt{tr((M.G).(M.G)^T)} = \sqrt{tr(M.G.G^T.M^T)}.$$

For a matrix $G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$, the square of the G -norm of our matrix, M , of second derivatives is

$$\|M\|_G^2 := (g_{11}x_{uu} + g_{12}x_{vu})^2 + (g_{21}x_{uu} + g_{22}x_{vu})^2 + (g_{11}x_{uv} + g_{12}x_{vv})^2 + (g_{21}x_{uv} + g_{22}x_{vv})^2.$$

If we now consider the functional defined by this norm

$$\int_{[0,1]^2} \|M\|_G^2 = \int_{[0,1]^2} (g_{11}^2 + g_{21}^2)\|x_{uu}\|^2 + (g_{12}^2 + g_{22}^2)\|x_{vv}\|^2 + (g_{11}^2 + g_{21}^2)\|x_{uv}\|^2 + (g_{12}^2 + g_{22}^2)\|x_{vu}\|^2 + 2(g_{11}g_{12} + g_{21}g_{22})(\langle x_{uu}, x_{vu} \rangle + \langle x_{uv}, x_{vv} \rangle) du dv,$$

then, the associated Euler-Lagrange equation is

$$0 = (g_{11}^2 + g_{21}^2)x_{uuuu} + (g_{12}^2 + g_{22}^2)x_{vvvv} + (g_{11}^2 + g_{21}^2 + g_{12}^2 + g_{22}^2)x_{uuvv} + 2(g_{11}g_{12} + g_{21}g_{22})(x_{uuuv} + x_{uvvv}).$$

Now by denoting $e_i = (g_{1i}, g_{2i})$, $i = 1, 2$, this equation can be written as

$$0 = \|e_1\|^2 x_{uuuu} + \|e_2\|^2 x_{vvvv} + (\|e_1\|^2 + \|e_2\|^2)x_{uuvv} + 2\langle e_1, e_2 \rangle (x_{uuuv} + x_{uvvv}),$$

and if we denote $\rho = \frac{\|e_1\|}{\|e_2\|}$ and $\langle e_1, e_2 \rangle = \|e_1\| \|e_2\| \cos t$, for some angle t , then it is equivalent to

$$0 = \rho^2 x_{uuuu} + 2\rho \cos t x_{uuuv} + (1 + \rho^2)x_{uuvv} + 2\rho \cos t x_{uvvv} + x_{vvvv}. \tag{1}$$

Let us consider the problem of finding a minimum of the functional

$$x_P \mapsto \int_{[0,1]^2} \|M\|_G^2 du dv,$$

among all polynomial patches with a prescribed boundary, being M the matrix of second derivatives of the Bézier patch, x_P , associated to the control net P .

This functional has a minimum in the Bézier case due to the following facts: First, it can be considered as a quadratic continuous real function of the interior control points. Second, since its definition comes from a norm it is bounded from below. Third, the infima is attained, if we restrict a continuous function to a compact subset we can affirm that the infima exists and that it is attained.

On the other hand if we prescribe four polynomial boundary curves and find the unique polynomial solution of the Euler-Lagrange PDE given in (1), as we will do in the following sections, then in addition we can assure that this PDE surface minimizes the associated functional.

Let us show some examples of functionals and the associated PDE surfaces.

Example 1. When $G = Id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then

$$\|M\|_{Id}^2 = \|\vec{x}_{uu}\|^2 + 2\|\vec{x}_{uv}\|^2 + \|\vec{x}_{vv}\|^2,$$

the functional is the Biharmonic functional and the associated Euler–Lagrange equation is the Biharmonic equation. Furthermore, if G is an orthogonal matrix, i.e., $G \cdot G^T = Id$, then $\|M\|_G = \|M\|_{Id}$, and the Euler–Lagrange equation is the same.

Example 2. When $G = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$ the square of the G -norm is

$$\|M\|_G^2 = \|x_{uu}\|^2 + (1 + \alpha^2)\|x_{uv}\|^2 + \alpha^2\|x_{vv}\|^2,$$

and the Euler–Lagrange equation associated to the functional defined by this norm is

$$x_{uuuu} + (1 + \alpha^2)x_{uuvv} + \alpha^2x_{vvvv} = 0.$$

In both cases, the two vectors, $\{e_1, e_2\}$, defining the matrix G are orthogonal.

Sometimes along this paper we will rewrite Eq. (1) as

$$A\vec{x}_{uuuu} + B\vec{x}_{uuvv} + C\vec{x}_{uuvv} + D\vec{x}_{uuvv} + E\vec{x}_{vvvv} = 0, \tag{2}$$

with $A = \rho^2$, $B = 2\rho \cos t$, $C = (1 + \rho^2)$, $D = 2\rho \cos t$, $E = 1$ (for $0 \leq t \leq 2\pi$).

When the two vectors, $\{e_1, e_2\}$, defining the matrix G are orthogonal then $B = D = 0$, we call it as the orthogonal case, otherwise we call it the non-orthogonal case, that is the general case, since it includes the orthogonal case.

In this paper we will compute the explicit solution of the general equation (1) and as a corollary we will consider the orthogonal case. Only for the sake of clarity we describe these two solution types in two separate sections, the general non-orthogonal case in Section 4 and the solution for the orthogonal case in Section 5.

3. From the boundary control points to the polynomial expression

Here we will compute Bézier polynomial solutions of the Euler–Lagrange equation (1) subject to given boundary conditions. Note that the Bézier polynomial solution we are looking for is of the form

$$\vec{x}(u, v) = \sum_{i,j=0}^n B_i^n(u)B_j^n(v)P_{ij}, \tag{3}$$

where $B_i^n(u)$ and $B_j^n(v)$ are n th order Bernstein polynomials and P_{ij} are the control points.

In Monterde and Ugail (2006) it was proved that the boundary control points of a Bézier solution of a general 4th-order PDE determine the first and second rows and columns of coefficients of its polynomial expression, and that from the first two rows and columns it can be determined all the coefficients a_{ij} .

Here we will describe, as in Monterde and Ugail (2006), the path from the boundary control points to the coefficients $a_{i0}, a_{0j}, a_{i1}, a_{1j}$. In the cited paper it was proved that given these four lines of coefficients a PDE surface that fulfills a general fourth order PDE can be determined, but here, in the following sections, we will compute the explicit solution of this general fourth order PDE in terms of four lines of coefficients.

Let us suppose that

$$\vec{x}(u, v) = \sum_{i,j=0}^n B_i^n(u)B_j^n(v)P_{ij} = \sum_{i,j=0}^n \frac{a_{ij}}{i!j!} u^i v^j, \tag{4}$$

where $a_{ij} \in \mathbf{R}^3$. First we will obtain the relation between the boundary control points $\{P_{0j}, P_{nj}, P_{i0}, P_{in}\}$ and the first row, $\{a_{i0}\}_{i=0}^n$, and the first column, $\{a_{0j}\}_{j=0}^n$, of coefficients $\{a_{ij}\}_{i,j=0}^n$.

Taking $u = 0$ in Eq. (4) we can obtain

$$\vec{x}(0, v) = \sum_{j=0}^n B_j^n(v)P_{0j} = \sum_{j=0}^n \frac{a_{0j}}{j!} v^j.$$

According to

$$B_j^n(v) = \binom{n}{j} v^j (1-v)^{n-j} = \binom{n}{j} \sum_{k=0}^{n-j} \binom{n-j}{k} (-1)^k v^{j+k} = \sum_{k=0}^{n-j} c_{j,k}^n v^{j+k},$$

where $c_{i,j}^n = (-1)^j \binom{n}{i} \binom{n-i}{j}$ we can say that

$$\bar{\mathbf{x}}(0, v) = \sum_{k=0}^n B_k^n(v) P_{0k} = \sum_{j=0}^n \left(\sum_{\ell=0}^j c_{\ell, j-\ell}^n P_{0\ell} \right) v^j,$$

and thus

$$a_{0j} = j! \left(\sum_{\ell=0}^j c_{\ell, j-\ell}^n P_{0\ell} \right), \quad j = 0, \dots, n. \tag{5}$$

Analogously, taking $v = 0$ in Eq. (4) we have

$$a_{i0} = i! \left(\sum_{\ell=0}^i c_{\ell, i-\ell}^n P_{\ell 0} \right), \quad i = 0, \dots, n. \tag{6}$$

For the second row, $\{a_{1j}\}_{j=0}^n$, and the second column $\{a_{i1}\}_{i=0}^n$, of coefficients we can apply a similar argument. Note that the coefficients a_{01} and a_{10} are already computed.

Taking $u = 1$ and $v = 1$ in Eq. (4) we have

$$\begin{aligned} \bar{\mathbf{x}}(1, v) &= \sum_{j=0}^n B_j^n(v) P_{nj} = \sum_{j=0}^n \frac{1}{j!} \left(\sum_{i=0}^n \frac{a_{ij}}{i!} \right) v^j, \\ \bar{\mathbf{x}}(u, 1) &= \sum_{i=0}^n B_i^n(u) P_{in} = \sum_{i=0}^n \frac{1}{i!} \left(\sum_{j=0}^n \frac{a_{ij}}{j!} \right) u^i. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{1}{j!} \left(\sum_{i=0}^n \frac{a_{ij}}{i!} \right) &= \left(\sum_{\ell=0}^j c_{\ell, j-\ell}^n P_{n\ell} \right), \\ \frac{1}{i!} \left(\sum_{j=0}^n \frac{a_{ij}}{j!} \right) &= \left(\sum_{\ell=0}^i c_{\ell, i-\ell}^n P_{\ell n} \right), \end{aligned}$$

and then

$$\begin{aligned} a_{1j} &= j! \left(\sum_{\ell=0}^j c_{\ell, j-\ell}^n P_{n\ell} \right) - a_{0j} - \sum_{i=2}^n \frac{a_{ij}}{i!}, \\ a_{i1} &= i! \left(\sum_{\ell=0}^i c_{\ell, i-\ell}^n P_{\ell n} \right) - a_{i0} - \sum_{j=2}^n \frac{a_{ij}}{j!}. \end{aligned} \tag{7}$$

This means that a_{1j} and a_{i1} can be computed from the boundary control points and coefficients $\{a_{ij}\}_{i>1}$ and $\{a_{ij}\}_{j>1}$. That is, the coefficient a_{1j} can be computed from the coefficients in column j in a row further down. These coefficients $\{a_{ij}\}_{i>1}$ are already known, see Monterde and Ugail (2006), since they belong to a diagonal row more to the right, note $i + j > 1 + j$. In the cited paper, it was described the algorithm to compute all coefficients a_{1j} of the prescribed boundary PDE surface, and analogously a_{i1} , recursively, starting from a_{1n} and a_{n1} and going downwards. Let us recall it now.

In terms of the coefficients a_{ij} , Eq. (2) is written as

$$0 = Aa_{k+4,\ell} + Ba_{k+3,\ell+1} + Ca_{k+2,\ell+2} + Da_{k+1,\ell+3} + Ea_{k,\ell+4}, \tag{8}$$

for all $k, \ell \in \mathbf{N}$.

In order to compute the coefficients in the polynomial expression of the solution one can recursively solve a set of systems of linear equations. Each system corresponds to a line parallel to the transverse diagonal of the following scheme.

$$\begin{array}{cccccccc}
 a_{00} & a_{01} & a_{02} & \dots & * & * & \mathbf{a_{0,k}} & \dots & a_{0,n-1} & a_{0,n} \\
 a_{10} & a_{11} & a_{12} & \dots & * & \mathbf{a_{1,k-1}} & * & \dots & a_{1,n-1} & a_{1,n} \\
 a_{20} & a_{21} & a_{22} & \dots & \mathbf{a_{2,k-2}} & * & * & \dots & a_{2,n-2} & 0 \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 * & * & \mathbf{a_{k-2,2}} & \dots & * & * & * & \dots & 0 & 0 \\
 * & \mathbf{a_{k-1,1}} & * & \dots & * & * & 0 & \dots & 0 & 0 \\
 \mathbf{a_{k,0}} & * & * & \dots & * & 0 & 0 & \dots & 0 & 0 \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 a_{n-1,0} & a_{n-1,1} & a_{n-1,2} & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
 a_{n,0} & a_{n,1} & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0
 \end{array}$$

Note that the subsystems involving $a_{k\ell}$ with $k + \ell > n + 1$ are homogeneous, then the only solution is the trivial one. This implies that $a_{k\ell} = 0$ if $k + \ell > n + 1$.

The first nonhomogeneous system corresponds to the line defined by $k + \ell = n + 1$. The coefficients $a_{0,n+1}$ and $a_{n+1,0}$ are zero, but the coefficients $a_{1,n}$ and $a_{n,1}$ can be computed using (7). Once the coefficients $a_{1,n}$ and $a_{n,1}$ are computed we can assume that the linear system for $k + \ell = n + 1$ has an associated coefficient matrix with nonvanishing determinant. In this case there is a unique solution and we can compute all the unknowns $a_{k,\ell}$ with $k + \ell = n + 1$.

We can now proceed with the line defined by $k + \ell = n$. Coefficients $a_{0,n}$ and $a_{n,0}$ can be computed in terms of control points using (5) and (6) while coefficients $a_{1,n-1}$ and $a_{n-1,1}$ are computed using (7). Note that $a_{n-1,2}$ and $a_{2,n-2}$, that have been computed in the previous step since they are more to the right, are needed to compute $a_{1,n-1}$ and $a_{n-1,1}$.

For the line corresponding to $k + \ell = m > 6$, the linear system to be solved is

$$\left\{ \begin{array}{l}
 -Aa_{m,0} - Ba_{m-1,1} = Ca_{m-2,2} + 2Da_{m-3,3} + Ea_{m-4,4}, \\
 -Aa_{m-1,1} = Ba_{m-2,2} + Ca_{m-3,3} + Da_{m-4,4} + Ea_{m-5,5}, \\
 0 = Aa_{m-2,2} + Ba_{m-3,3} + Ca_{m-4,4} + Da_{m-5,5} + Ea_{m-6,6}, \\
 \dots \\
 0 = Aa_{6,m-6} + Ba_{5,m-5} + Ca_{4,m-4} + Da_{3,m-3} + Ea_{2,m-2}, \\
 -Ea_{1,m-1} = Aa_{5,m-5} + Ba_{4,m-4} + Ca_{3,m-3} + Da_{2,m-2}, \\
 -Ea_{0,m} - Da_{1,m-1} = Aa_{4,m-4} + Ba_{3,m-3} + Ca_{2,m-2}.
 \end{array} \right. \tag{9}$$

As we said, the procedure described above was given in Monterde and Ugail (2006) where we discussed in detail the solvability of the linear systems. We note that they can be solved numerically, but the existence of an explicit solution, for $A = \rho^2$, $B = 2\rho \cos t$, $C = 1 + \rho^2$, $D = 2\rho \cos t$ and $E = 1$, that is our current goal, makes the algorithm even more efficient.

4. The explicit solution of the general case

In this section we study the explicit solution of the more general case which we call the non-orthogonal case, where the coefficients of the Euler–Lagrange equation (2) are

$$A = \rho^2, \quad B = 2\rho \cos t, \quad C = 1 + \rho^2, \quad D = 2\rho \cos t, \quad E = 1.$$

Before we outline the explicit solution scheme for this case we discuss some preliminaries.

We know that given one boundary curve and the first three transversal partial derivatives along it, that is the values of f and its first three partial derivatives f_u, f_{uu}, f_{uuu} , along $(0, v)$ a solution of $\Delta^2 f = 0$ is uniquely determined, see Monterde and Ugail (2004). Thus, in terms of the coefficients, if the first four rows of coefficients are known, then all the other coefficients of a Biharmonic surface can be determined. Analogously if the first four columns of coefficients were the given data.

In the following lemma we will establish a similar result, but we do not prove that a PDE surface that fulfills Eq. (2) is entirely determined by four lines of coefficients, we give instead an explicit formula of such a surface.

Specifically, here we give the explicit solution in terms of the first four columns of coefficients, that is, given the boundary curve $\bar{\mathbf{x}}(u, 0)$ and its first three transversal partial derivatives we describe the associated PDE surface in the following lemma.

Lemma 1. *The solution of the linear system*

$$0 = \rho^2 a_{k+4,\ell} + 2\rho \cos t a_{k+3,\ell+1} + (1 + \rho^2) a_{k+2,\ell+2} + 2\rho \cos t a_{k+1,\ell+3} + a_{k,\ell+4}, \tag{10}$$

for all $k, \ell \in \mathbb{N}$ in terms of the first four columns of coefficients is given by

$$a_{k,\ell} = A_\ell a_{k+\ell,0} + B_\ell a_{k+\ell-1,1} + C_\ell a_{k+\ell-2,2} + D_\ell a_{k+\ell-3,3}, \quad \ell > 3, \tag{11}$$

where

$$\begin{aligned}
 A_n &= -\rho^2 D_{n-1}, & B_n &= \sin \frac{n\pi}{2} + D_n, & C_n &= -\cos \frac{n\pi}{2} - \rho^2 D_{n-1}, \\
 D_n &= \frac{(\rho^2 - 1) \sin \frac{n\pi}{2} - 2\rho \cos t \cos \frac{n\pi}{2} + (-\rho)^{n-1} \operatorname{csct}(\rho^2 \sin(n-2)t + \sin nt)}{1 + \rho^4 + 2\rho^2 \cos 2t}.
 \end{aligned}
 \tag{12}$$

Proof. The proof involves a computation by substituting solution (11) in Eq. (10). Here we check to see that each one of the four sequences defined by the coefficients, $\{A_\ell\}_{\ell \geq 0}$, $\{B_\ell\}_{\ell \geq 0}$, $\{C_\ell\}_{\ell \geq 0}$ and $\{D_\ell\}_{\ell \geq 0}$, verifies the following recurrence relation

$$\rho^2 x_\ell + 2\rho \cos t x_{\ell+1} + (1 + \rho^2) x_{\ell+2} + 2\rho \cos t x_{\ell+3} + x_{\ell+4} = 0.$$

In order to compute the general term of the recurrence sequence we have to solve the quintic equation

$$\rho^2 + 2\rho \cos t x + (1 + \rho^2) x^2 + 2\rho \cos t x^3 + x^4 = 0.$$

This equation can be written as

$$(1 + x^2)(\rho^2 + 2\rho \cos t x + x^2) = 0.$$

Therefore, its solutions are, $\pm i$, $-\rho e^{\pm it}$ and the general term is

$$x_n = A i^n + B (-i)^n + C (-\rho)^n e^{nit} + D (-\rho)^n e^{-nit},$$

where the parameters A, B, C and D must be computed from the four initial terms of the recurrence sequence.

For example, in order to compute the general term of the sequence $\{A_\ell\}_{\ell \geq 0}$ we have to put the initial conditions, also called boundary conditions in the terminology of recurrence sequences

$$A_0 = 1, \quad A_1 = 0, \quad A_2 = 0, \quad A_3 = 0.$$

For the general term of the sequence $\{B_\ell\}_{\ell \geq 0}$, the initial conditions are

$$B_0 = 0, \quad B_1 = 1, \quad B_2 = 0, \quad B_3 = 0$$

and so on.

Finally, straightforward computations give us the explicit solution (12). \square

Now, the procedure we adopt here is to interchange the third and fourth columns by the first two rows. We have to solve the linear system,

$$\begin{aligned}
 a_{0,k+\ell} &= A_{k+\ell} a_{k+\ell,0} + B_{k+\ell} a_{k+\ell,-1,1} + C_{k+\ell} a_{k+\ell,-2,2} + D_{k+\ell} a_{k+\ell,-3,3}, \\
 a_{1,k+\ell-1} &= A_{k+\ell-1} a_{k+\ell,0} + B_{k+\ell-1} a_{k+\ell,-1,1} + C_{k+\ell-1} a_{k+\ell,-2,2} + D_{k+\ell-1} a_{k+\ell,-3,3},
 \end{aligned}$$

taking as unknowns $a_{k+\ell,-2,2}, a_{k+\ell,-3,3}$. Then we substitute the solution into the expression (11).

Proposition 2. The solution of the linear system in Eq. (10) for all $k, \ell \in \mathbb{N}$ in terms of the first two rows and the first two columns of coefficients is given by

$$a_{k,\ell} := \mathbf{A}_{k,\ell} a_{k+\ell,0} + \mathbf{B}_{k,\ell} a_{k+\ell,-1,1} + \mathbf{C}_{k,\ell} a_{1,k+\ell-1} + \mathbf{D}_{k,\ell} a_{0,k+\ell} \tag{13}$$

for all $k, \ell > 1$, where

$$\begin{aligned}
 \mathbf{A}_{k,\ell} &= \frac{1}{M_{k+\ell}(C, D)} (A_\ell M_{k+\ell}(C, D) - C_\ell M_{k+\ell}(A, D) + D_\ell M_{k+\ell}(A, C)), \\
 \mathbf{B}_{k,\ell} &= \frac{1}{M_{k+\ell}(C, D)} (B_\ell M_{k+\ell}(C, D) - C_\ell M_{k+\ell}(B, D) + D_\ell M_{k+\ell}(B, C)), \\
 \mathbf{C}_{k,\ell} &= \frac{1}{M_{k+\ell}(C, D)} (-C_\ell D_{k+\ell} + D_\ell C_{k+\ell}), \\
 \mathbf{D}_{k,\ell} &= \frac{1}{M_{k+\ell}(C, D)} (C_\ell D_{k+\ell-1} - D_\ell C_{k+\ell-1}),
 \end{aligned}$$

where, $A_\ell, B_\ell, C_\ell, D_\ell$ are defined in Eq. (12) and

$$M_n(C, D) = \det \begin{pmatrix} C_n & D_n \\ C_{n-1} & D_{n-1} \end{pmatrix}.$$

5. The explicit solution of the orthogonal case

The aim here is to study some important particular cases (e.g. Coons, Harmonic and Biharmonic) for which the determinant of the system coefficient matrix is different from zero. Here we will obtain the explicit solution of all the linear systems involved for the cases involving Coons, standard Biharmonic and modified Biharmonic equations. In order to give the explicit polynomial solution to these important cases we first consider Eq. (2) with $B = D = 0$ and for any value of A, C and E .

Although, as we said, the orthogonal case is a particular case of the non-orthogonal PDE, we haven't deduced here the corresponding result as a consequence of the non-orthogonal case, in fact, we keep the notation A, B, C, D, E in some places because we solve the orthogonal PDE:

$$0 = Ax_{uuuu} + Cx_{uuvv} + Ex_{vvvv}$$

for any value of A, C and E . After that, we give some specific values to A, C and E , and, as a particular case, we deduce the solution of the orthogonal case, with $A = \rho^2, C = 1 + \rho^2, E = 1$ and $B = D = 0$.

5.1. The case $B = D = 0$

The linear system (9) has the following structure. For the line corresponding to $k + \ell = m > 6$, the linear system to be solved is

$$\begin{cases} -Aa_{m,0} = Ca_{m-2,2} + Ea_{m-4,4}, \\ -Aa_{m-1,1} = Ca_{m-3,3} + Ea_{m-5,5}, \\ 0 = Aa_{m-2,2} + Ca_{m-4,4} + Ea_{m-6,6}, \\ \dots \\ 0 = Aa_{6,m-6} + Ca_{4,m-4} + Ea_{2,m-2}, \\ -Ea_{1,m-1} = Aa_{5,m-5} + Ca_{3,m-3}, \\ -Ea_{0,m} = Aa_{4,m-4} + Ca_{2,m-2}. \end{cases} \tag{14}$$

Note that half of the equations deals with coefficients

$$\{a_{m,0}, a_{m-2,2}, a_{m-4,4}, a_{m-6,6}, \dots\}$$

while the other half deals with the other coefficients

$$\{a_{m-1,1}, a_{m-3,3}, a_{m-5,5}, a_{m-7,7}, \dots\}.$$

We can, therefore, split the linear system into two similar linear systems, both of them with the same matrix of coefficients, the Toeplitz tridiagonal matrix

$$T_n = \begin{pmatrix} C & E & 0 & \dots & 0 & 0 \\ A & C & E & \dots & 0 & 0 \\ 0 & A & C & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & C & E \\ 0 & 0 & 0 & \dots & A & C \end{pmatrix}. \tag{15}$$

Some well-known results on tridiagonal Toeplitz matrices must be recalled. The first one is that the determinant of these scalar tridiagonal matrices verifies the following recurrence relation: suppose we denote $G_n = \det(T_n)$, then

$$G_{n+2} = CG_{n+1} - EAG_n. \tag{16}$$

Moreover, from this recurrence equation it is possible to compute the general term of the sequence $\{G_n\}_{n=1}^\infty$. The explicit expression of the general term depends on the multiplicity of the roots of the associated characteristic equation $x^2 - Cx + EA = 0$.

If the characteristic equation has two simple roots, $x_0 \neq x_1$, then for $n > 0$,

$$G_n = \frac{x_0^{n+1} - x_1^{n+1}}{x_0 - x_1}. \tag{17}$$

If it has one double root, x_0 , then

$$G_n = (n + 1)x_0^n. \tag{18}$$

Finally, as a convention we take $G_n = 0$ for $n \leq 0$.

Lemma 3. The solution of the linear system,

$$0 = Ab_{k+2,\ell} + Cb_{k+1,\ell+1} + Eb_{k,\ell+2}, \tag{19}$$

for all $k, \ell \in \mathbb{N}$ is given by

$$b_{k,\ell} = \frac{G_{k-1}(-A)^\ell b_{k+\ell,0} + G_{\ell-1}(-E)^k b_{0,k+\ell}}{G_{k+\ell-1}},$$

where, $G_n = \det(T_n)$ which can be computed using (17) or (18) depending on the roots of (16).

Proof. Let us compute

$$\begin{aligned} & Ab_{k+2,\ell} + Cb_{k+1,\ell+1} + Eb_{k,\ell+2} \\ &= A \frac{G_{k+1}(-A)^\ell b_{k+2+\ell,0} + G_{\ell-1}(-E)^{k+2} b_{0,k+2+\ell}}{G_{k+\ell+1}} + C \frac{G_k(-A)^{\ell+1} b_{k+1+\ell+1,0} + G_\ell(-E)^{k+1} b_{0,k+1+\ell+1}}{G_{k+\ell+1}} \\ & \quad + E \frac{G_{k-1}(-A)^{\ell+2} b_{k+\ell+2,0} + G_{\ell+1}(-E)^k b_{0,k+\ell+2}}{G_{k+\ell+1}} \\ &= \frac{1}{G_{k+\ell+1}} ((AG_{k+1}(-A)^\ell + CG_k(-A)^{\ell+1} + EG_{k-1}(-A)^{\ell+2}) b_{k+\ell+2,0} \\ & \quad + (AG_{\ell-1}(-E)^{k+2} + CG_\ell(-E)^{k+1} + EG_{\ell+1}(-E)^k) b_{0,k+\ell+2}) \\ &= \frac{(-A)^{\ell+1}}{G_{k+\ell+1}} (G_{k+1} - CG_k + EAG_{k-1}) b_{k+\ell+2,0} + \frac{(-E)^{k+1}}{G_{k+\ell+1}} (AEG_{\ell-1} - CG_\ell + G_{\ell+1}) b_{0,k+\ell+2}. \end{aligned}$$

Finally note that due to the recurrence equation (16) we have

$$\begin{aligned} G_{k+1} - CG_k + EAG_{k-1} &= 0, \\ AEG_{\ell-1} - CG_\ell + G_{\ell+1} &= 0. \quad \square \end{aligned}$$

With the results of the above lemma, all we have to do is to apply it to the two linear systems arising from Eq. (14).

Proposition 4. The solution of the linear system,

$$0 = Aa_{k+4,\ell} + Ca_{k+2,\ell+2} + Ea_{k,\ell+4}, \tag{20}$$

for all $k, \ell \in \mathbb{N}$ is given by

$$a_{k,\ell} := \frac{G_{\lfloor \frac{k}{2} \rfloor - 1} (-A)^{\lfloor \frac{\ell}{2} \rfloor} a_{k+2\lfloor \frac{\ell}{2} \rfloor, \ell \bmod 2} + G_{\lfloor \frac{\ell}{2} \rfloor - 1} (-E)^{\lfloor \frac{k}{2} \rfloor} a_{k \bmod 2, 2\lfloor \frac{k}{2} \rfloor + \ell}}{G_{\lfloor \frac{k}{2} \rfloor + \lfloor \frac{\ell}{2} \rfloor - 1}}. \tag{21}$$

This equation lets us to determine all $a_{k,\ell}$ given the first two rows and columns of coefficients. Moreover, note that right-hand term in the expression (21) reduces to the correct values when k or ℓ are equal to 0 or 1. This is due to the fact that $G_n = 0$ for $n \leq 0$. Therefore, expression (21) is valid for all $k, \ell \geq 0$.

Finally, in this section we present some more particular cases.

5.2. Some important cases and the orthogonal case

5.2.1. The Coons case

In this case, Coons solution is well known (Sabin, 1994). When $A = B = D = E = 0$ and $C \neq 0$ the PDE is reduced to $x_{uvv} = 0$, and the associated linear equations are reduced to $a_{k,\ell} = 0$ for $k, \ell > 1$. The other coefficients are directly computed from Eqs. (5), (6), and (7). That is,

$$a_{1j} = j! \left(\sum_{\ell=0}^j c_{\ell,j-\ell}^n P_{n\ell} \right) - a_{0j}, \quad j > 0,$$

and

$$a_{i1} = i! \left(\sum_{\ell=0}^i c_{\ell,i-\ell}^n P_{n\ell} \right) - a_{i0}, \quad i > 0.$$

From the above equations we can easily obtain the well-known solution of $x_{uuvv} = 0$ with a fixed boundary given by the curves $\bar{\mathbf{x}}(u, 0)$, $\bar{\mathbf{x}}(u, 1)$, $\bar{\mathbf{x}}(0, v)$, $\bar{\mathbf{x}}(1, v)$. The solution is

$$\bar{\mathbf{x}}(u, v) = (1 - u)\bar{\mathbf{x}}(0, v) + u\bar{\mathbf{x}}(1, v) + (1 - v)\bar{\mathbf{x}}(u, 0) + v\bar{\mathbf{x}}(u, 1) + (1 - u - v) \begin{pmatrix} \bar{\mathbf{x}}(0, 0) & \bar{\mathbf{x}}(0, 1) \\ \bar{\mathbf{x}}(1, 0) & \bar{\mathbf{x}}(1, 1) \end{pmatrix} \begin{pmatrix} 1 - v \\ v \end{pmatrix}.$$

5.2.2. The modified Biharmonic case

Here we find the explicit solution of the modified Biharmonic equation

$$\bar{\mathbf{x}}_{uuuu} + 2\alpha^2\bar{\mathbf{x}}_{uuvv} + \alpha^4\bar{\mathbf{x}}_{vvvv} = 0,$$

where $\alpha \in \mathbb{R}$. In this case, $A = 1$, $B = 0$, $C = 2\alpha^2$, $D = 0$ and $E = \alpha^4$. The solution has the following form. For $k, \ell > 1$,

$$a_{k,\ell} := \frac{1}{\lfloor \frac{k}{2} \rfloor + \lfloor \frac{\ell}{2} \rfloor} \left(\left(-\frac{1}{\alpha^2} \right)^{\lfloor \frac{\ell}{2} \rfloor} \lfloor \frac{k}{2} \rfloor a_{k+2\lfloor \frac{\ell}{2} \rfloor, \ell \bmod 2} + (-\alpha^2)^{\lfloor \frac{\ell}{2} \rfloor} \lfloor \frac{\ell}{2} \rfloor a_{k \bmod 2, 2\lfloor \frac{k}{2} \rfloor + \ell} \right).$$

5.2.3. The Biharmonic case

This case is a particular case of the previous one with $a = 1$. It was studied in Monterde and Ugail (2004). There we only proved the existence of the solutions. We will include it here for the sake of completeness and since we can now provide explicit solution for this case.

The explicit solution has the following form. For $k, \ell > 1$,

$$a_{k,\ell} := \frac{1}{\lfloor \frac{k}{2} \rfloor + \lfloor \frac{\ell}{2} \rfloor} \left((-1)^{\lfloor \frac{\ell}{2} \rfloor} \lfloor \frac{k}{2} \rfloor a_{k+2\lfloor \frac{\ell}{2} \rfloor, \ell \bmod 2} + (-1)^{\lfloor \frac{k}{2} \rfloor} \lfloor \frac{\ell}{2} \rfloor a_{k \bmod 2, 2\lfloor \frac{k}{2} \rfloor + \ell} \right).$$

5.2.4. The orthogonal case

Here we find the explicit solution of Eq. (1) with $t = \frac{\pi}{2}$

$$\rho^2 x_{uuuu} + (1 + \rho^2)x_{uuvv} + x_{vvvv} = 0$$

where $\rho \in \mathbb{R}$, $\rho \neq 1$. In this case, $A = \rho^2$, $B = 0$, $C = 1 + \rho^2$, $D = 0$ and $E = 1$. The solution has the following form. For $k, \ell > 1$,

$$a_{k,\ell} := \frac{(-1)^{\lfloor \frac{k}{2} \rfloor} (\rho^{2\lfloor \frac{\ell}{2} \rfloor} - 1) a_{k \bmod 2, 2\lfloor \frac{k}{2} \rfloor + \ell} + (-\rho^2)^{\lfloor \frac{\ell}{2} \rfloor} (\rho^{2\lfloor \frac{k}{2} \rfloor} - 1) a_{k+2\lfloor \frac{\ell}{2} \rfloor, \ell \bmod 2}}{\rho^{2(\lfloor \frac{k}{2} \rfloor + \lfloor \frac{\ell}{2} \rfloor)} - 1}.$$

6. From the polynomial expression to the boundary control points

Up to now we can assure that the knowledge of the boundary control points implies the knowledge of the first two rows and columns of coefficients, and then the knowledge of all the coefficients, i.e. the whole PDE surface.

The coefficients a_{i0} and a_{0j} in terms of control points are given in Eqs. (5) and (6), from Eq. (7) and Proposition 2, it can be computed the coefficients a_{1j} and a_{i1} of the prescribed boundary PDE surface recursively, starting from a_{1n} and a_{n1} and going downwards,

$$a_{1j} = j! \sum_{\ell=0}^j c_{\ell, j-\ell}^n P_{n\ell} - a_{0j} - \sum_{i=2}^n \frac{\mathbf{A}_{i,j} a_{i+j,0} + \mathbf{B}_{i,j} a_{i+j-1,1} + \mathbf{C}_{i,j} a_{1,i+j-1} + \mathbf{D}_{i,j} a_{0,i+j}}{i!},$$

$$a_{i1} = i! \sum_{\ell=0}^i c_{\ell, i-\ell}^n P_{n\ell} - a_{i0} - \sum_{j=2}^n \frac{\mathbf{A}_{i,j} a_{i+j,0} + \mathbf{B}_{i,j} a_{i+j-1,1} + \mathbf{C}_{i,j} a_{1,i+j-1} + \mathbf{D}_{i,j} a_{0,i+j}}{j!}.$$

Therefore, from the boundary data, the two first rows and columns of coefficients in the polynomial expression of the PDE surface can be determined, and then Proposition 2 lets the computation of all the coefficients, i.e. the whole PDE surface,

$$a_{k,\ell} := \mathbf{A}_{k,\ell} a_{k+\ell,0} + \mathbf{B}_{k,\ell} a_{k+\ell-1,1} + \mathbf{C}_{k,\ell} a_{1,k+\ell-1} + \mathbf{D}_{k,\ell} a_{0,k+\ell}.$$

Finally, once obtained the $a_{k,\ell}$ coefficients of the polynomial expression we can come back, if it is wanted, to the Bézier form. The control points of the PDE Bézier surface are given by

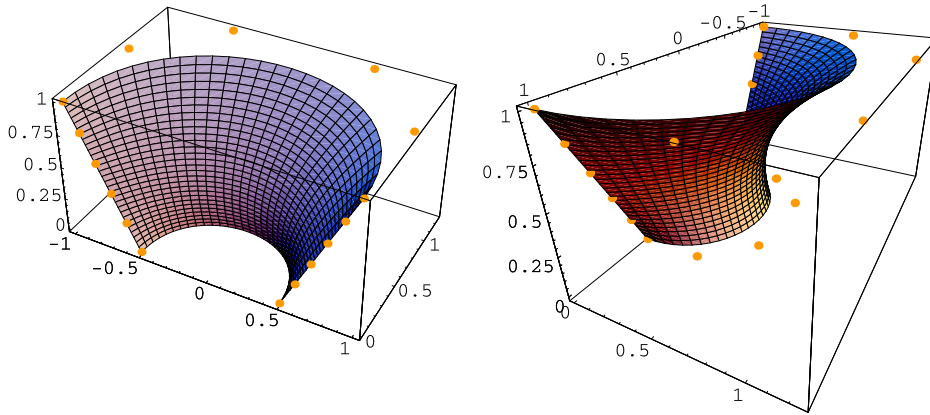


Fig. 1. Two views of a typical surface solution plot arising from Biharmonic equation showing the control points of boundary curves defined as Bézier curves of degree 5.

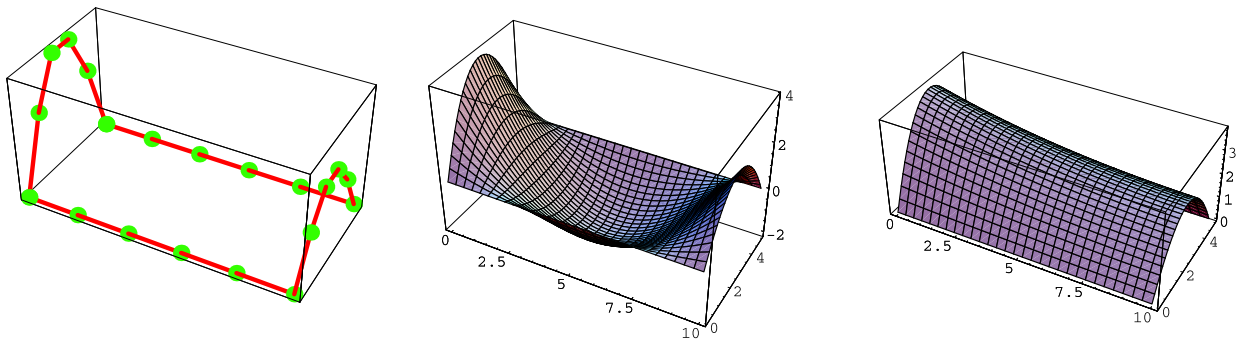


Fig. 2. Surface solution plot arising from the modified Biharmonic equation. Left: the boundary configuration. Middle: surface solution plot with $a^2 = 0.1$. Right: surface solution plot with $a^2 = 1.5$.

$$P_{i,j} = \sum_{k,\ell=0}^{i,j} \frac{\binom{i}{k} \binom{j}{\ell} a_{k\ell}}{\binom{n}{k} \binom{n}{\ell} k!\ell!},$$

for all $i, j = 0, \dots, n$.

The knowledge of the boundary control points implies the knowledge of the whole control net, the PDE surface is totally determined.

7. Examples

In this section we discuss a series of examples relating to the explicit Bézier solution methodology for the PDEs we have described. In particular we discuss several examples relating to surface generation with a given boundary configuration both in the orthogonal and non-orthogonal cases. We also compare the speed of our explicit solution scheme with solution arising from directly solving the associated linear system.

Fig. 1 shows a typical surface solution plot arising from Biharmonic equation where the four boundary conditions are given as Bézier curves of degree 5.

Fig. 2 shows the surface solution plot arising from modified Biharmonic equation with different values a . The four boundary conditions that are given as Bézier curves of degree 5 are also shown.

In order to compare the analytic solution scheme described here, we have compared the CPU times it takes to compute the surface through the explicit solution scheme and the scheme involving the numerical solution of systems of linear equations described in Monterde and Ugail (2006). The computations were carried out for the same surface shown in Fig. 1 but for different polynomial degrees. In all cases a parametric mesh of size 50×50 was used. For the numerical computation of the linear systems we have utilized the LU decomposition algorithm outlined in Numerical Recipes (Press et al., 1992). The computations were performed on a laptop computer with 1.73 GHz CPU with 512 MB RAM and the results are shown in Table 1. From these results is it evident that in all cases the explicit solution scheme outperforms the numerical scheme from the point of view of computational efficiency. This is expected since the explicit solution scheme only involves computing analytic functions, e.g. factorials and powers, whereas in the case of the numerical scheme a systems of linear

Table 1
Comparison of explicit solution scheme with the scheme involving a system of linear equations. (Time in ms.)

Polynomial degree	Explicit solution	Solution involving linear systems
4	0.78	1.12
5	0.93	1.94
8	1.11	2.83
10	1.87	3.12
12	4.89	11.11
15	8.48	15.49
18	11.41	22.91
20	15.42	28.75

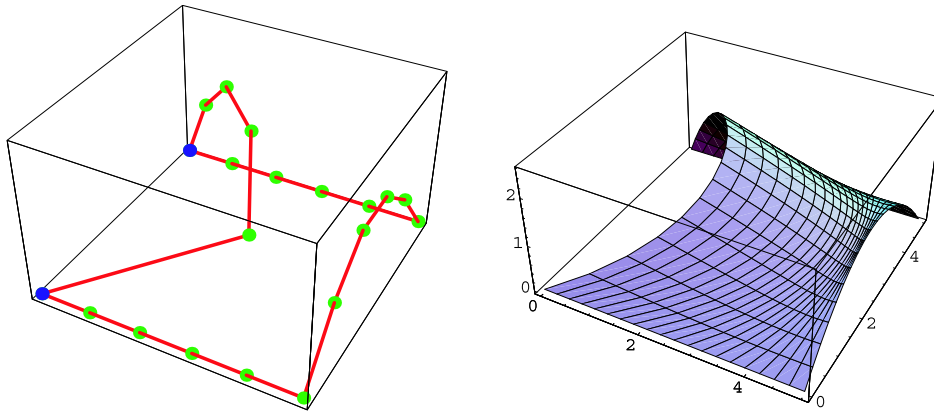


Fig. 3. Left: The boundary control points and the common boundary curves. Right: Draw of the Biharmonic solution (for an angle $t = \frac{\pi}{2}$) showing how coordinate lines present a bad configuration.

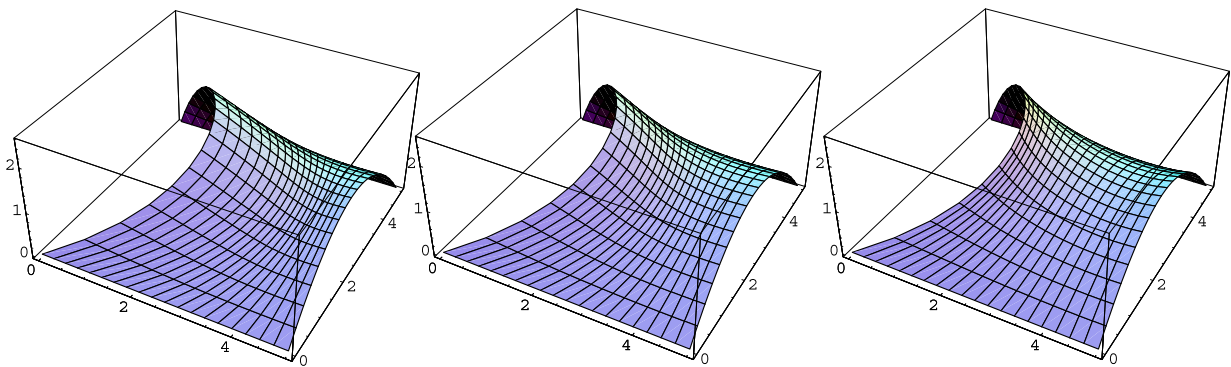


Fig. 4. Three Bézier surfaces, with the same boundary configuration arising from the solution of the non-orthogonal equation for different values of the angle ($t = \pi - \frac{20}{3}, \pi - \frac{3}{4}, \pi - \frac{7}{8}$).

equations are solved over the domain to approximate the solution. Note in both cases the speed of computation depends directly on the degree, i.e. number of control points, of the polynomials chosen to represent the boundary conditions.

Fig. 3 shows an example of Bézier surface obtained by means of Proposition 2 along with the boundary configurations. For this example the value of the parameter is $t = \frac{\pi}{2}$. One way to generate different surface configuration is via the use of the parameter t . For instance, in this example, for the left boundary curve one could observe that the corner points where the Bézier surface passes through is not to be isothermal, i.e., the tangent vectors to the coordinates lines at that corners are not orthogonal and of different length. Thus, when the boundary configuration is not too regular the solutions of the Biharmonic equation does not generate a uniform mesh. However, the use of the angle parameter in the non-orthogonal equation can improve the shape of the resulting surfaces. This is illustrated in the examples given in Fig. 4 where different values for the t have been utilized to generate surfaces from the same boundary configuration.

8. Conclusion

In this paper we present an explicit polynomial solution method for surface generation which is characterized by some boundary configuration whereby the resulting surface conforms to fourth order linear elliptic PDEs. In particular, the paper deals with surfaces generated as explicit Bézier polynomial solutions. A variety of examples relating to surface generation from a given boundary configuration is discussed.

Since the problem we are addressing here is essentially solving a boundary-value problem associated with the linear Biharmonic equation we presume that the results we present here have much wider implication than simply to boundary based smooth surfaces generation. As one would note that the solution of the Biharmonic equation is related to solving problems in a number of areas of science and engineering, including elasticity and slow fluid flow. Several explicit solution approaches in the past has been adopted to the boundary value problem associated with the Biharmonic equation. Of those techniques, since the equation is linear, it is usual to seek separable solutions appropriate to the given geometry and then superpose these to satisfy the given boundary data. This approach, for example, has been adopted by Bloor and Wilson in their earlier developments the PDE method (Bloor and Wilson, 1990).

As mentioned earlier, in this paper we are considering an explicit solution approach based on polynomial functions. Thus, the method we propose in this paper can be utilized for solving a fourth order PDE in the restricted domain of the polynomial functions, the PDE being the Euler–Lagrange equation of a quadratic functional defined by a norm. Therefore, the PDE surface we obtain minimizes, in addition, the associated functional.

The solution method we have discussed has direct relevance to surface generation from boundary data as well as mesh generation from a given boundary configuration for engineering analysis purposes. We wish to pursue our research in these directions in the future.

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