The total graph of a hypergraph¹

Peter Cowling *

Université Libre de Bruxelles, Service de Mathématiques de la Gestion, CP 210/01, Boulevard du Triomphe, 1050 Bruxelles, Belgium

Received 7 July 1995; revised 4 December 1995

Abstract

Let \( H \) be a hypergraph with vertices \( V(H) \) and hyperedges \( E(H) \). The total graph of \( H \), \( T(H) \), is the simple graph with vertex set \( V(H) \cup E(H) \) where vertices \( x \) and \( y \) of \( T(H) \) are adjacent if and only if \( x \) is contained in, contains or is adjacent to \( y \) in \( H \). We give a simple characterisation of those graphs which are the total graphs of some hypergraphs. We show that the total graph uniquely defines a linear hypergraph up to isomorphism and duality and present examples to show that this is not the case for general nonlinear hypergraphs. We give a polynomial time algorithm for the problem of deciding whether a given graph is the total graph of a linear hypergraph.

1. Introduction

A hypergraph \( H \) is a pair \( (V(H), E(H)) \), where \( V(H) \) is a finite set of vertices and \( E(H) \) is a finite family of nonempty subsets of \( V(H) \) called hyperedges or just edges, with \( \bigcup_{E \in E(H)} E = V(H) \). \( H \) is linear if for all distinct \( E, E' \in E(H) \), \( |E \cap E'| \leq 1 \), so for a linear hypergraph there may be no repeated hyperedges of cardinality greater than one. Distinct vertices \( v, v' \in V(H) \) are adjacent if there is some hyperedge \( E \in E(H) \) with \( v, v' \in E \). Distinct hyperedges \( E, E' \in E(H) \) are adjacent if \( E \cap E' \neq \emptyset \). Vertex \( v \in V(H) \) is incident with hyperedge \( E \in E(H) \), and vice versa, if \( v \in E \). A path from \( v \in V(H) \) to \( v' \in V(H) \) is a finite sequence \( (v, E_1, w_1, E_2, w_2, \ldots, E_{k-1}, w_{k-1}, E_k, v') \) such that \( v \in E_1, w_i \in E_i \cap E_{i+1} \) for \( i = 1, 2, \ldots, k - 1 \) and \( v' \in E_k \). \( H \) is connected if for every pair \( v, v' \in V(H) \) there is some path from \( v \) to \( v' \).

The dual of \( H = (\{v_1, v_2, \ldots, v_n\}, [E_1, E_2, \ldots, E_m]) \), \( H^* \), is the hypergraph whose vertices \( \{e_1, e_2, \ldots, e_m\} \) correspond to the hyperedges of \( H \), and with hyperedges

\[
V_i = \{e_j : v_i \in E_j \text{ in } H\} \quad (i = 1, 2, \ldots, n).
\]

¹ Research supported by the SERC.
* E-mail: cowling@ulb.ac.be.
The rank of $H$, $\text{rank}(H)$, is the maximum cardinality of a hyperedge in $E(H)$. A hyperedge of rank one is a loop. The degree of a vertex $v \in V(H)$, $\text{deg}_H(v)$, is the number of hyperedges containing $v$. The maximum degree among vertices of $H$ is denoted $\Delta(H)$.

A simple graph is a linear hypergraph of rank 2 without loops. A multigraph with loops is a hypergraph of rank 2. Note that under our definition these graphs may not have isolated vertices. When talking of the edge $\{v, w\}$ of a graph, we will often write simply $vw$. The subgraph of graph $G = (V(G), E(G))$ induced by $W \subseteq V(G)$ is the graph with vertex set $W$ containing only those edges of $E \subseteq E(G)$ with $E \subseteq W$. The distance from $v \in V(G)$ to $w \in V(G)$, $d_G(v, w)$, is the minimum number of edges in a path from $v$ to $w$. The neighbour set of a vertex $v \in V(G)$, $N_G(v)$, is the set of vertices at distance 1 from $v$. The closed neighbour set of $v$, $\overline{N}_G(v)$, is $N_G(v) \cup \{v\}$. A bipartite graph is a simple graph $G$ whose vertex set $V(G)$ has a bipartition $(S, T)$ such that $S$ and $T$ both induce a graph with no edges. The complete graph on $n$ vertices, $K_n$, is a simple graph on $n$ vertices where every pair of vertices is adjacent. The cycle on $n$ vertices $C_n$ is a connected simple graph on $n$ vertices where every vertex has degree 2. Two graphs $G_1$, $G_2$ are isomorphic (written $G_1 \cong G_2$) if there is some bijection $\theta : V(G_1) \to V(G_2)$ such that $vw \in E(G_1)$ if and only if $\theta(v)\theta(w) \in E(G_2)$. Note $C_n \cong C_n^*$. The cycles are the only family of graphs for which this is the case. Note that the dual $G^*$ of a graph $G$ is not a graph unless $G$ has maximum degree 2.

Associated with the hypergraph $H$ we have several graphs. The 2-section of $H$, $H_2$, is the simple graph with vertex set $V(H)$ where distinct $x, y \in V(H)$ are adjacent in $H_2$ if and only if they are adjacent in $H$. The line graph of $H$, $L(H)$, is the simple graph with vertex set $E(H)$ where distinct $E, E' \in E(H)$ are adjacent in $L(H)$ if and only if $E$ and $E'$ are adjacent in $H$. Then $L(H) \cong (H^*)_2$. The incidence graph of $H$, $I(H)$, is the bipartite graph with vertices $V(H) \cup E(H)$ and bipartition $(V(H), E(H))$ where $v \in V(H)$ is adjacent to $E \in E(H)$ if and only if $v$ is contained in the hyperedge $E$ of $H$. Then $I(H) \cong I(H^*)$ and $I(H)$ uniquely defines $H$ up to isomorphism and duality. Two hypergraphs $H_1$, $H_2$ are isomorphic (written $H_1 \cong H_2$) if $I(H_1) \cong I(H_2)$ and the isomorphism $\theta$ maps $V(H_1)$ onto $V(H_2)$ and maps $E(H_1)$ onto $E(H_2)$. $H_1$ is dual isomorphic to $H_2$ if $H_1 \cong H_2^*$. The total graph of $H$, $T(H)$, is the simple graph with vertices $V(H) \cup E(H)$ where $x, y \in (V(H) \cup E(H))$ are adjacent if and only if $x$ is contained in, contains or is adjacent to $y$ in $H$. The edge set of $T(H)$ is the disjoint union of the edge sets of $H_2$, $L(H)$ and $I(H)$ and we have thus $T(H) \cong T(H^*)$. The middle graph of $H$, $M(H)$ (see [5,13]) is the subgraph of $T(H)$ formed by deleting all edges connecting pairs of vertices of $V(H)$.

A (strong) vertex colouring of hypergraph $H$ is a mapping $C : V(H) \to \{1, 2, \ldots, k\}$ such that every pair of adjacent vertices receives different colours. The smallest $k$ for which a vertex colouring exists is the chromatic number $\chi(H)$. For all hypergraphs $H$ we have that $\chi(H) = \chi(H_2)$. A total colouring of $H$ is a mapping $C : (V(H) \cup E(H)) \to \{1, 2, \ldots, k_r\}$ such that every pair of adjacent vertices, every pair of adjacent hyperedges
and every incident vertex and hyperedge receive different colours. The smallest $k$ for which such a colouring exists is the total chromatic number $\chi_t(H)$. Note that a total colouring of $H$ defines a total colouring of $H^*$, hence $\chi_t(H) = \chi_t(H^*)$. This 'self-duality' is one of the most useful properties of total colourings of hypergraphs, which we will use repeatedly in this paper. The total graph of a hypergraph arises since for all hypergraphs $H$, $\chi_t(H) = \chi(T(H))$.

The study of the total chromatic number for hypergraphs and in particular linear hypergraphs, is motivated in part by the total colouring conjecture, posed independently by Behzad [1] and Vizing [14], which we now give.

**Total colouring conjecture** (Behzad [1] and Vizing [14]). Let $G$ be a simple graph. Then

$$\chi_t(G) \leq \Delta(G) + 2.$$ 

A stronger conjecture for hypergraphs was given in [8].

**Total colouring conjecture for hypergraphs.** Let $H$ be a linear hypergraph without loops or vertices of degree one. Then

$$\chi_t(H) \leq \min\{\Delta(H_2), \Delta(L(H))\} + 2.$$ 

Evidence for the total colouring conjecture for graphs has been gathered in two principle ways, first by proving the conjecture true for a wide range of classes of graphs and secondly by bounding the total chromatic number for all graphs. A recent survey is given in [11]. In [6,12] results are proved about total chromatic numbers of specific classes of hypergraphs. Upper bounds on the total chromatic number of all hypergraphs are given in [8].

Considering the total graph allows us to reformulate these conjectures in terms of the chromatic number of a simple graph. The total graph of a simple graph has been considered in several earlier papers and shown to possess a rich combinatorial structure. In [3] Behzad and Radjavi show that the total graph of a simple graph $G$ uniquely defines the graph, up to isomorphism. Other properties are considered in [2,4] and other papers. Gavril [9] uses ideas from these papers to produce a polynomial time algorithm for the problem of recognising whether a simple graph is the total graph of another simple graph.

In [5] it is shown that every hypergraph is uniquely defined by its middle graph, up to isomorphism. These ideas are used in [13] to produce a polynomial time algorithm for the problem of deciding whether a given simple graph is the middle graph of some hypergraph.

In the theory of radio frequency assignment [10,15] we encounter the $L(a_1,a_2)$ colouring paradigm for simple graphs. Given a simple graph $G$ we wish to find a colouring $\phi : V(G) \rightarrow \{1,2,...,k\}$ such that for adjacent vertices $v_1,w_1$ we have $|\phi(v_1) - \phi(w_1)| \geq a_1$ and for vertices $v_2,w_2$ which are distance 2 apart, we have $|\phi(v_2) - \phi(w_2)| \geq a_2$. Then a total colouring of hypergraph $H$ corresponds to an $L(1,1)$
colouring of $I(H)$ and the total graph $T(H)$ corresponds to the graph obtained from $I(H)$ by adding edges joining all vertices which are distance 2 apart.

The aim of this paper is to determine whether the total graph of a hypergraph uniquely defines the hypergraph up to isomorphism and duality. The main result of the paper is Theorem 4.

**Theorem 4.** Let $H$ be a connected linear hypergraph. Then for arbitrary (possibly nonlinear) hypergraph $H'$, $T(H') \cong T(H) \iff H' \cong H$ or $H' \cong H^*$.

A similar result does not hold for general hypergraphs. We give, in Section 6, an example of a simple graph $G_1$ which is the total graph of two nonisomorphic and nondual-isomorphic hypergraphs $H_1$ and $H_2$.

In order to arrive at the theorem, we must first prove results about the structure of the total graph of a hypergraph. In Section 2 we show that a simple property of graph $G$, the total partition property, can be used to determine whether $G$ is the total graph of some hypergraph. In Section 3 we investigate graphs with this property, showing that we need only determine the partition of $G$ locally in order to uniquely specify the partition over the whole of $G$. In Section 4 we investigate the possible ways in which two different partitions may interact. We are then ready, in Section 5 to prove Theorem 4. In Section 6 we give an example which shows that the total graph $G$ does not necessarily uniquely define a hypergraph unless $G$ is the total graph of a linear hypergraph. In Section 7 we present a polynomial time algorithm for the problem of deciding whether a given graph is the total graph of a linear hypergraph. We present conclusions and open problems in Section 8.

### 2. The total partition property

We characterise total graphs of hypergraphs as those possessing the total partition property. Throughout the paper we will develop several theorems concerning graphs with this simple property.

**Definition 1.** Let $G$ be a simple graph. Let $(\mathcal{R}, \mathcal{B})$ be a partition of $V(G)$. Define the equivalence relation $\sim$ with $x \sim y \iff x$ and $y$ belong to the same block of the partition. Then we say that $(\mathcal{R}, \mathcal{B})$ has the total partition property if for all distinct $v, w, x \in V(G)$

$$vw \in E(G) \text{ with } v \sim w \iff \exists x \text{ with } vx, wx \in E(G) \text{ and } x \not\sim v \sim w.$$  

Throughout the paper, when we refer to the total partition $(\mathcal{R}, \mathcal{B})$ of $V(G)$, it will be convenient to consider $(\mathcal{R}, \mathcal{B})$ as a two-colouring of the vertices of $G$, where the vertices of $\mathcal{R}$ are coloured red and the vertices of $\mathcal{B}$ are coloured blue.
The importance of the total partition property is given in the next theorem, where we show that there is a 1–1 correspondence between total partitions of \( V(G) \) and hypergraphs \( H \) such that \( T(H) \cong G \).

**Theorem 1.** Let \( G \) be a simple graph. \( G \) is the total graph of some hypergraph \( H \) if and only if there is some partition of \( V(G) \) satisfying the total partition property.

**Proof.** Let \( H = (\mathcal{V}, \mathcal{E}) \) be a hypergraph, and let \( G = T(H) \) be the total graph of \( H \), with vertex set \( V(G) = \mathcal{V} \cup \mathcal{E} \). We show that \((\mathcal{V}, \mathcal{E})\) is a total partition of \( G \). Consider \( v_1, v_2 \in \mathcal{V}, v_1v_2 \in E(G) \iff v_1 \text{ and } v_2 \text{ both lie in some hyperedge } e \in \mathcal{E} \text{ in } H \iff \exists e \in \mathcal{E} \text{ with } v_1e, v_2e \in E(G) \). Similarly, \( f_1, f_2 \in \mathcal{V}, f_1f_2 \in E(G) \iff f_1 \text{ and } f_2 \text{ both contain some vertex } w \in \mathcal{V} \text{ in } H \iff \exists w \in \mathcal{V} \text{ with } f_1w, f_2w \in E(G) \).

Now consider that we have some graph \( G \) with total partition \((\mathcal{R}, \mathcal{B})\). Let \( H \) be the hypergraph with vertex set \( \mathcal{R} \) and hyperedge set \( \mathcal{B} \) whose incidence graph \( I(H) \) is given by the subgraph of \( G \) containing only edges going from vertices in \( \mathcal{R} \) to vertices in \( \mathcal{B} \). Note that \( H \) is well defined by any graph \( G \), whether or not \( G \) is total. Then it is easily shown that vertices of \( \mathcal{R} \) induce a subgraph in \( G \) which is isomorphic to \( T_2 \) and vertices of \( \mathcal{B} \) induce a subgraph which is isomorphic to \( L(H) \), thus \( T(H) \cong G \).

If the partition \((\mathcal{R}, \mathcal{B})\) of \( V(G) \) has the total partition property then it corresponds to a hypergraph \( H \) with vertex set \( \mathcal{R} \) and hyperedge set \( \mathcal{B} \) where \( I(H) \) is the bipartite subgraph of \( G \) obtained by deleting all those edges which connect pairs of vertices of the same colour. We say that \( H \) is the hypergraph induced by the partition \((\mathcal{R}, \mathcal{B})\) of \( V(G) \). Of course, \((\mathcal{R}, \mathcal{B})\) has the total partition property if and only if \((\mathcal{B}, \mathcal{R})\) has. In fact, if \((\mathcal{R}, \mathcal{B})\) is the total partition of \( G \) which induces hypergraph \( H \), then \((\mathcal{B}, \mathcal{R})\) will induce the dual \( H^* \).

To characterise the total graphs of the linear hypergraphs have only to change our definition above very slightly.

**Proposition 1.** Let \( G \) be a simple graph. \( G \) is the total graph of some linear hypergraph \( \iff \) there is some partition \((\mathcal{R}, \mathcal{B})\) of \( V(G) \) (with corresponding equivalence relation \( \sim \)) satisfying the total partition property, such that for every pair of adjacent vertices \( v, w \) in \( V(G) \) with \( v \sim w \) there is exactly one vertex \( x \) with \( x \sim v \sim w \).

**Proof.** Stating that each pair \( v, w \) of adjacent vertices with \( v \sim w \) has exactly one neighbour \( x \) with \( x \sim v \sim w \) is equivalent to the assertion that in the hypergraph induced by \((\mathcal{R}, \mathcal{B})\), adjacent vertices \( v \) and \( w \) are contained in exactly one hyperedge, or adjacent hyperedges \( v \) and \( w \) have exactly one vertex in common. This is the definition of linearity for \( H \).
We will say that a partition \((\mathcal{A}, \mathcal{B})\) of \(V(G)\) as in Proposition 1 has the linear total partition property, or is a linear total partition.

Note that the class of hypergraphs and the class of linear hypergraphs are both closed under hypergraph duality. Since the class of graphs is not closed under this operation, no such simple characterisation exists for graphs. In our partition of the total graph of a graph one of the classes must be specialised to be vertices of the resulting graph (the special vertices of Behzad [2]) and one must be specialised to be edges (the nonspecial vertices of Behzad [2]).

**Proposition 2.** Let \(G\) be a simple graph. \(G\) is the total graph of a multigraph with loops \(\Leftrightarrow\) there is some partition \(\{\mathcal{A}, \mathcal{B}\}\) of \(V(G)\) satisfying the total partition property, such that for all vertices \(y \in \mathcal{B}\), \(y\) is adjacent to at most two vertices from \(\mathcal{A}\).

**Proposition 3.** Let \(G\) be a simple graph. \(G\) is the total graph of simple graph \(\Leftrightarrow\) there is some linear total partition \(\{\mathcal{A}, \mathcal{B}\}\) of \(V(G)\) such that for all vertices \(y \in \mathcal{B}\), \(y\) is adjacent to exactly two vertices from \(\mathcal{A}\).

3. **Properties**

First we will need to introduce some notation which we will use throughout the rest of the paper. Let \(G\) be a simple, connected graph. Let \(v\) be an arbitrary fixed vertex of \(G\). Consider two partitions of \(V(G)\), \((\mathcal{A}, \mathcal{B})\) and \((\mathcal{C}, \mathcal{D})\), each with the total partition property. We assume that \(v \in \mathcal{A} \cap \mathcal{C}\), without loss of generality, since for any given vertex we may exchange \(\mathcal{A}, \mathcal{B}\) and/or \(\mathcal{C}, \mathcal{D}\) so that this is the case. Then we define:

\[
D_i = \{x \in V(G) : d_G(v,x) = i\} \quad (i = 0, 1, \ldots, k),
\]

\[
R_i = D_i \cap \mathcal{A} \quad (i = 0, 1, \ldots, k),
\]

\[
B_i = D_i \cap \mathcal{B} \quad (i = 0, 1, \ldots, k),
\]

\[
V_i = D_i \cap \mathcal{C} \quad (i = 0, 1, \ldots, k),
\]

\[
E_i = D_i \cap \mathcal{D} \quad (i = 0, 1, \ldots, k),
\]

where \(k = \max_{w \in V(G)} d_G(v,w)\). So we have \(D_0 = R_0 = V_0 = \{v\}, B_0 = E_0 = \emptyset\).

We have immediately a property which will be useful later.

**Lemma 1.** Let \((\mathcal{A}, \mathcal{B})\) be a partition of the vertices of simple graph \(G\) with the total partition property, and define \(v, B_i, R_i\) and \(D_i\) as above, then vertices of \(B_1\) induce a clique in \(G\).
Proof. For any pair of vertices $x, y \in B_1$, $x$ and $y$ are both adjacent to $v \in R_0$. Hence $x$ and $y$ must be adjacent in $G$. □

Theorem 2, which is an immediate corollary of the next proposition, shows us that a total partition defined in the neighbourhood of any vertex will uniquely define the total partition over the whole graph. This generalises Theorem 2 of Behzad [2] and the fact that our proof is somewhat easier arises from the added simplicity of considering the general class of total partitions of an arbitrary graph, rather than the specialised class of the total partitions of the total graph of a simple graph.

Proposition 4. Let $(\mathcal{A}, \mathcal{B})$ be a partition of the vertices of simple, connected graph $G$, with the total partition property, and define $v, B_i, R_i$ and $D_i$ as above, then for $x \in D_i$,

1. $x \in B_i \iff x$ is adjacent to some vertex of $B_{i-1}$ ($i = 2, 3, \ldots$),
2. $x \in R_i \iff x$ is not adjacent to any vertex of $B_{i-1}$ ($i = 2, 3, \ldots$),
3. $x \in B_i \Rightarrow x$ is adjacent to some vertex of $R_{i-1}$ ($i = 1, 2, \ldots$),
4. $x \in R_i \Rightarrow x$ is adjacent to some vertex of $B_i$ ($i = 1, 2, \ldots$).

Proof. The proof is by induction on $i$, the distance from vertex $v$. First note that any vertex of $B_1$ is adjacent to $v \in R_0$, hence (3) is true for $i = 1$. Now, for any $x_1 \in R_1$, adjacent vertices $x_1$ and $v$ are both red. Thus they must have a common blue neighbour. This neighbour must lie in $B_1$. Hence (4) is true for $i = 1$.

Now suppose for induction that (1) and (2) are true for $i = 2, 3, \ldots, k - 1$ and that (3) and (4) are true for $i = 1, 2, \ldots, k - 1$ for some $k \geq 2$ (for $k = 2$ this means that we suppose only that (3) and (4) are true for $i = 1$).

Consider $x_2 \in D_k$ which is adjacent to some $y_2 \in B_{k-1}$. By (3) (for $i = k - 1$), $y_2$ has some neighbour $z_2 \in R_{k-2}$. If $x_2$ were in $R_k$ then red $x_2$ and red $z_2$ would both be adjacent to blue $y_2$, but $x_2 \in D_k$ and $z_2 \in D_{k-2}$ cannot be adjacent. Hence $x_2$ must be in $B_k$. Conversely, for $x_3 \in R_k$, $x_3$ is not adjacent to any vertex of $B_{k-1}$.

Consider $x_4 \in D_k$ which is not adjacent to any vertex of $B_{k-1}$. Then $x_4$ is adjacent to some $y_4 \in R_{k-1}$ and by (4) (for $i = k - 1$), $y_4$ is adjacent to some $z_4 \in B_{k-1}$. If $x_4$ were blue then blue $x_4$ and blue $z_4$ would both be adjacent to red $y_4$, so $x_4$ would be adjacent to $z_4$, contradicting the hypothesis that $x_4$ is not adjacent to any vertex of $B_{k-1}$. Hence $x_4 \in R_k$. Conversely, for $x_5 \in B_k$, $x_5$ is adjacent to some vertex of $B_{k-1}$. Hence (1) and (2) are true for $i = k$.

Consider $x_6 \in B_k$. Then $x_6$ has some neighbour $y_6 \in B_{k-1}$ by (1) (for $i = k$). $x_6$ and $y_6$ must have some common red neighbour $z_6$ and $z_6$ must be in $R_{k-1}$ since no vertex of $R_k$ can be adjacent to $y_6 \in B_{k-1}$ by (2) (for $i = k$). Hence (3) is true for $i = k$.

Consider $x_7 \in R_k$. Then $x_7$ has some neighbour $y_7 \in R_{k-1}$ since $x_7$ can have no neighbour in $B_{k-1}$ by (2) (for $i = k$). $x_7$ and $y_7$ must have a common blue neighbour $z_7$, which must be in $B_k$, since $x_7 \in R_k$ is not adjacent to any vertex of $B_{k-1}$ by (2) (for $i = k$). Hence (4) is true for $i = k$ and the proof by induction is complete. □
In fact Proposition 4 gives us immediately a method TOTALEXTEND for generating a total partition, given \( v, R_1 \) and \( B_1 \), as follows:

Subroutine TOTALEXTEND\((v, R_1, B_1, G)\).

\[
\begin{align*}
i &= 2 \\
\text{while } (D_i \neq \emptyset): \\
&\{ \\
&\quad \text{for all vertices } x \in D_i \\
&\quad \quad \{ \\
&\quad \quad \quad \text{if } x \text{ is adjacent to some vertex of } B_{i-1} \\
&\quad \quad \quad \quad B_i \leftarrow B_i \cup \{x\} \\
&\quad \quad \quad \text{else} \\
&\quad \quad \quad \quad R_i \leftarrow R_i \cup \{x\} \\
&\quad \quad \} \\
&\quad i \leftarrow i + 1 \\
&\}\end{align*}
\]

This subroutine proceeds by breadth-first search. Its time complexity is \( O(m) \), where \( m \) is the number of edges of \( G \).

We have as an immediate corollary

**Theorem 2.** Let \( G \) be a connected simple graph and let \( v \in V(G) \). If the colours of the vertices in \( \overline{N}_G(v) \) are specified, i.e. \( R_1 \) and \( B_1 \) are specified, then there is at most one bipartition of \( V(G) \) satisfying the total partition property.

If we consider the hypergraph \( H \) induced by the total partition \((\mathcal{R}, \mathcal{B})\), then Theorem 2 says that we need only specify that a given vertex \( v \) of \( G \) corresponds to a hyperedge [vertex] of \( H \) and further specify which of the vertices of \( N(v) \) correspond to the vertices [hyperedges] of \( H \) contained in [containing] this hyperedge [vertex].

Proposition 4 tells us that if any class \( D_i \) received only one colour, then all vertices of \( D_{i+1}, D_{i+2}, \ldots \) would receive only this colour. Clearly this is not possible and, indeed, we have

**Proposition 5.** Let \( G \) be a simple graph. Let \((\mathcal{R}, \mathcal{B})\) be a partition of the vertices of \( G \) with the total partition property. Then we have that for all vertices \( w \) of \( G \) which belong to a connected component of \( G \) which is not a complete graph, \( w \) is adjacent to vertices of both \( \mathcal{R} \) and \( \mathcal{B} \).

**Proof.** Consider, without loss of generality, that \( w \in \mathcal{R} \). \( w \) cannot be only adjacent to vertices of \( \mathcal{R} \), since if \( w \) is adjacent to some \( x \in \mathcal{R} \) then \( w \) and \( x \) must have a common neighbour in \( \mathcal{B} \). Suppose for contradiction that \( w \) were only adjacent to vertices of \( \mathcal{B} \). By Proposition 4(1) this would mean that all vertices of the connected component in which \( w \) lies apart from \( w \) are blue. Then if there were some vertex \( y \) at distance 2
from \( w \), \( y \) must be adjacent to some vertex \( x \) at distance 1 from \( w \), but \( x \) and \( y \) do not have a common red neighbour. Hence there is no vertex at distance 2 from \( w \), and the connected component in which \( w \) lies is a clique, by Lemma 1, contradicting the hypothesis. □

Note that \( G \) may not have isolated vertices. Any larger clique of \( G \) may be coloured arbitrarily so long as the clique contains vertices of both \( R \) and \( B \).

4. Recolourings

Suppose we have a simple graph \( G \), which has two distinct total partitions \((\mathcal{V}', \mathcal{E}')\) and \((\mathcal{R}, \mathcal{B})\), in the sense that \((\mathcal{V}', \mathcal{E}') \neq (\mathcal{R}, \mathcal{B})\) and \((\mathcal{V}', \mathcal{E}') \neq (\mathcal{B}, \mathcal{R})\). Then \((\mathcal{V}', \mathcal{E}')\) and \((\mathcal{R}, \mathcal{B})\) must be different in the neighbourhood of each vertex of \( G \), by Theorem 2. In this section we will explore some of the local behaviours exhibited by two different colourings in the neighbourhood of a vertex in the case that one of the partitions has the linear total partition property. In [7] we explore other results about the interaction of the two partitions in the case where neither is linear.

In the four lemmas which follow, let \( G \) be a simple connected graph, which is not a complete graph. Define \( v, (\mathcal{R}, \mathcal{B}), (\mathcal{V}', \mathcal{E}), R_i, B_i, V_i \) and \( E_i \) as in the previous section. Hence we have that \( v \in \mathcal{V} \cap \mathcal{R} \). Assume that \((\mathcal{V}', \mathcal{E}')\) is a linear total partition of \( V(G) \).

The structural results which we prove in the next four lemmas have rather involved proofs, which the reader may wish to skip on a first reading. They will enable us to prove the main result of the paper in the next section.

There are four possible ways in which \( v, E_1, R_1 \) and \( B_1 \) may interact. We will consider these four in turn.

Lemma 2. If \( B_1 = E_1 \), \( R_1 = V_1 \) then \((\mathcal{V}', \mathcal{E}') = (\mathcal{R}, \mathcal{B})\).

Proof. This is simply a restatement of Theorem 2. □

Lemma 3. It is not possible that \( B_1 \) is a proper subset of \( E_1 \).

Proof. Assume for contradiction that \( B_1 \) is a proper subset of \( E_1 \). Let \( e \) be a vertex of \( E_1 \cap R_1 \), which must exist by hypothesis. Now consider some \( w \) in \( V_1 = V_1 \cap R_1 \), which must exists by Proposition 5. \( w \) has some neighbour \( f \) in \( B_1 = E_1 \cap B_1 \) by Proposition 4(4) applied to \((\mathcal{R}, \mathcal{B})\). \( f \) is adjacent to \( e \) by lemma 1, so \( f \in \mathcal{B} \) is adjacent to \( e, w \in \mathcal{R} \), thus \( e \) and \( w \) must be adjacent. Then \( e, f \in \mathcal{E} \) are both adjacent to both of \( v, w \in \mathcal{V}' \), which contradicts the hypothesis that \((\mathcal{V}', \mathcal{E}')\) is a linear total partition of \( V(G) \). □
Later we will show that it is also not possible for $B_1$ to be a proper superset of $E_1$. Note that in this case we would have $|B_1 \cap V_1| - |B_1 \cap E_1| > 0$.

**Lemma 4.** Suppose that $B_1$ is a nonempty subset of $V_1$ (and $R_1$ contains $E_1$), then we have the following:

1. No vertex of $V_1 \cap R_1$ is adjacent to any vertex of $\mathcal{E} \cap \mathcal{B}$.
2. Every $w \in V_1 \cap B_1$ is adjacent to exactly one $e \in \mathcal{E} \cap \mathcal{B}$ which is in $E_1 \cap R_1$.
3. Every $e \in E_1 \cap R_1$ is adjacent to exactly one $w \in \mathcal{E} \cap \mathcal{B}$ which is in $V_1 \cap B_1$.

**Proof.** No $w_1 \in V_1 \cap R_1$ is adjacent to any $e_1 \in E_2 \cap B_2$. Suppose for contradiction that $w_1 \in V_1 \cap R_1$ and $e_1 \in E_2 \cap B_2$ are adjacent. We know $w_1$ is adjacent to some $f_1 \in E_1$ (so $f_1 \in E_1 \cap R_1$) by Proposition 4(4) applied to $(\mathcal{V}, \mathcal{E})$. Also $f_1 \in R_1$ is adjacent to some $x_1 \in V_1 \cap B_1$ by Proposition 4(4) applied to $(\mathcal{R}, \mathcal{B})$. $f_1 \in \mathcal{E}$ and $e_1 \in \mathcal{E}$ are both adjacent to $w_1 \in \mathcal{V}$, so $e_1$ and $f_1$ must be adjacent. Now $e_1, x_1 \in \mathcal{B}$ are both adjacent to $f_1 \in \mathcal{R}$, but if $e_1$ and $x_1$ are adjacent then $e_1, f_1 \in \mathcal{E}$ are both adjacent to both of $w_1, x_1 \in \mathcal{V}$, contradicting linearity of $(\mathcal{V}, \mathcal{E})$. Since $E_1 \cap B_1 = \emptyset$ we have that no vertex of $V_1 \cap R_1$ is adjacent to any vertex of $\mathcal{E} \cap \mathcal{B}$. Hence part 1 is proved.

Every vertex of $V_1 \cap B_1$ has at least one neighbour in $E_1 \cap R_1$, by Proposition 4(4) applied to the partition $(\mathcal{V}, \mathcal{E})$. Further, no vertex of $V_1 \cap B_1$ is adjacent to more than one vertex of $E_1 \cap R_1$ since if $w_2 \in V_1 \cap B_1$ were to be adjacent to the pair $e_2, f_2 \in E_1 \cap R_1$ then $e_2, f_2 \in \mathcal{V}$ would both be adjacent to both of $e_2, f_2 \in \mathcal{E}$ and this would contradict linearity of $(\mathcal{V}, \mathcal{E})$. Thus every vertex of $V_1 \cap B_1$ has exactly one neighbour in $E_1 \cap R_1$. By Proposition 4(2) applied to $(\mathcal{R}, \mathcal{B})$, no vertex of $R_2$ can be adjacent to a vertex of $B_1$. Thus each vertex of $V_1 \cap B_1$ has exactly one neighbour in $\mathcal{E} \cap \mathcal{B}$, which is in $E_1 \cap R_1$. Hence part 2 is proved.

Every vertex in $E_1 \cap R_1$ has at least one neighbour in $V_1 \cap B_1$, by Proposition 4(4) applied to the partition $(\mathcal{R}, \mathcal{B})$. No vertex of $E_1$ has a neighbour in $V_2$, by Proposition 4(2) applied to $(\mathcal{V}, \mathcal{E})$. Hence all neighbours of vertices in $E_1 \cap R_1$ which lie in $\mathcal{V} \cap \mathcal{B}$ must lie in $V_1 \cap B_1$.

We will prove that each vertex of $E_1 \cap R_1$ has exactly one neighbour in $V_1 \cap B_1$ in two stages. First we prove that for each $e_3 \in E_1 \cap R_1$, $e_3$ has a neighbour $w_3 \in V_1 \cap B_1$ such that $e_3$ and $w_3$ have a common neighbour in $E_2 \cap B_2$. If $|E_1 \cap R_1| > 1$ then consider vertex $f_3 \in (E_1 \cap R_1) - \{e_3\}$. $e_3$ and $f_3$ are adjacent by Lemma 1, so they must have a common neighbour in $\mathcal{B}$. This common neighbour $g_3$ is not in $V_1 \cap B_1$, since we have shown that each vertex of $V_1 \cap B_1$ has exactly one neighbour in $E_1 \cap R_1$, so $g_3$ must lie in $B_2$. Further, since $g_3$ is in $D_2$ and adjacent to $e_3 \in E_1$, we must have $g_3 \in E_2 \cap B_2$. Then $e_3$ and $g_3$ must have some common neighbour $w_3 \in \mathcal{V}$, which must be in $V_1$ since no vertex of $V_2$ can be adjacent to $e_3 \in E_1$ by Proposition 4(2) applied to $(\mathcal{V}, \mathcal{E})$. Thus we have that $w_3 \in V_1 \cap B_1$ since we have already shown that no vertex of $V_1 \cap R_1$ can be adjacent to a vertex of $E_2 \cap B_2$.

If $|E_1 \cap R_1| = 1$ then $e_4 \in E_1 \cap R_1$ must have some neighbour $g_4$ in $E_2$, by Proposition 5. If $g_4 \in E_2 \cap B_2$ then as above we find that $e_4, g_4$ have a common neighbour in $V_1 \cap B_1$. Suppose $g_4 \in E_2 \cap R_2$. Then $g_4, e_4 \in \mathcal{B}$ must have some common neighbour $h_4 \in \mathcal{B}$.
$h_4$ must be in $B_2$, since $g_4 \in R_2$ cannot be adjacent to any vertex of $B_1$, by Proposition 4(2). Furthermore, $h_4$ must be in $E_2$, since it is adjacent to $e_4 \in E_1$, by Proposition 4(1). Hence $e_4$ has some neighbour $h_4 \in E_2 \cap B_2$ and $e_4$ and $h_4$ must have some common neighbour in $V_1 \cap B_1$ as above.

Now we show that each $e_5 \in E_1 \cap R_1$ is adjacent to exactly one $w_5 \in V_1 \cap B_1$. Suppose that some $e_5 \in E_1$ is adjacent to two distinct vertices $w_5, x_5 \in V_1 \cap B_1$ where we may assume that $e_5$ and $w_5$ have common neighbour $g_5 \in E_2 \cap B_2$ by the above. Then $g_5, x_5 \in \mathcal{B}$ are both adjacent to $e_5 \in \mathcal{B}$, but if $g_5$ and $x_5$ are adjacent, then $e_5, g_5 \in \mathcal{E}$ are both adjacent to both of $w_5, x_5 \in \mathcal{V}$, contradicting linearity of $(\mathcal{V}, \mathcal{G})$.

**Lemma 5.** Suppose that $|B_1 \cap V_1| |B_1 \cap E_1| > 0$. In this case we must have:

1. $B_1 = \{e, w_1, w_2, \ldots, w_k\}$, where $e \in E_1$, $w_1, w_2, \ldots, w_k \in V_1$ and $w_1, w_2, \ldots, w_k$ are all the neighbours of $e$ in $V_1$.
2. Each vertex of $V_1 \cap B_1$ is not adjacent to any vertex of $\mathcal{E} \cap \mathcal{B}$.
3. Each vertex of $E_1 \cap R_1$ is not adjacent to any vertex of $\mathcal{V} \cap \mathcal{B}$. $E_1 \cap R_1 \neq \emptyset$.
4. All vertices of $V_1 \cap R_1$ have exactly one neighbour in $\mathcal{E} \cap \mathcal{B}$, which is in $E_2 \cap B_2$.

**Proof.** We must have $|B_1 \cap E_1| = 1$. Suppose for contradiction that there are two distinct vertices $e_1, f_1 \in E_1 \cap B_1$ and note that there is some $x_1 \in V_1 \cap B_1$ by hypothesis. By Lemma 1, $\{v, x_1, e_1, f_1\}$ induces a clique in $G$. Then $e_1, f_1 \in \mathcal{E}$ are both be adjacent to both of $v, x_1 \in \mathcal{V}$, contradicting linearity of $(\mathcal{V}, \mathcal{G})$.

Let $e$ be the unique vertex in $B_1 \cap E_1$ for the remainder of the proof.

We must have $E_1 \cap R_1 \neq \emptyset$. Suppose for contradiction that $E_1 = \{e\}$. We have by Proposition 5 that $e$ has some neighbour $f_2 \in E_2$ and by Proposition 4(1) applied to $(\mathcal{B}, \mathcal{B})$ we have that $f_2 \in E_2 \cap B_2$. $e, f_2 \in \mathcal{B}$ must have some common neighbour $x_2 \in \mathcal{B}$ and by Proposition 4(3) we must have $x_2 \in V_1 \cap R_1$. Now there is some $y_2 \in V_1 \cap B_1$ by hypothesis. By Proposition 4(4) applied to $(\mathcal{V}, \mathcal{G})$, $x_2, y_2 \in \mathcal{V}$ are both adjacent to $e \in \mathcal{E}$, thus $x_2$ and $y_2$ are adjacent. Since $y_2, f_2 \in \mathcal{B}$ are both adjacent to $x_2 \in \mathcal{B}$ we must have that $f_2$ and $y_2$ are adjacent. Then both of $x_2, y_2 \in \mathcal{V}$ are adjacent to both of $e, f_2 \in \mathcal{E}$, contradicting linearity of $(\mathcal{V}, \mathcal{G})$.

Any vertex in $V_1$ which is not adjacent to $e$ must be in $V_1 \cap R_1$. Suppose for contradiction that $x_3 \in V_1 \cap B_1$ is not adjacent to $e$. By Proposition 4(4), $x_3$ must have some neighbour $f_3 \in E_1 \cap R_1$, where $f_1$ is adjacent to $e$ by Lemma 1. Then $x_3 \in \mathcal{B}$ and $e \in \mathcal{B}$ are both adjacent to $f_3 \in \mathcal{B}$, contradicting the hypothesis that $e$ and $x_3$ are nonadjacent.

All vertices of $V_1$ adjacent to $e$ must be in $V_1 \cap B_1$. Assume for contradiction that there is some $x_4 \in (V_1 \cap R_1)$ adjacent to $e$. There is some $f_4 \in E_1 \cap R_1$ which is adjacent to $e$ by Lemma 1. Now $f_4, x_4 \in \mathcal{B}$ and both are adjacent to $e \in \mathcal{B}$, however, if $f_4$ and $x_4$ are adjacent then $e, f_4 \in \mathcal{E}$ are both adjacent to both of $x_4, v \in \mathcal{V}$, contradicting linearity of $(\mathcal{V}, \mathcal{G})$. Hence part 1 is proved.

Each $f_5 \in E_1 \cap R_1$ cannot be adjacent to any $x_5 \in V_1 \cap B_1$ since otherwise $e, f_5 \in \mathcal{E}$ would both be adjacent to both of $v, x_5 \in \mathcal{V}$, contradicting linearity of $(\mathcal{V}, \mathcal{G})$. Since
$x_5 \in V_1 \cap B_1$ cannot be adjacent to any vertex of $R_2$ by Proposition 4(2) applied to $(\mathcal{R}, \mathcal{B})$, we have that no $x_5 \in V_1 \cap B_1$ is adjacent to any vertex of $\mathcal{R} \cap \mathcal{R}$ and part 2 is proved. Further, since no $f_5 \in E_1 \cap R_1$ is adjacent to any vertex of $V_2$ by Proposition 4(2) applied to $(\mathcal{V}, \mathcal{E})$, no $f_5 \in E_1 \cap R_1$ is adjacent to any vertex of $\mathcal{V} \cap \mathcal{B}$ and part 3 is proved.

Any $x_6 \in V_1 \cap R_1$ has at least one neighbour in $E_2 \cap B_2$. By Theorem 4(4), $x_6$ is adjacent to some vertex $f_6 \in E_1$. By part 1 above, we must have $f_6 \in E_1 \cap R_1$. Now $x_6, f_6 \in \mathcal{R}$ must have a common neighbour $g_6 \in \mathcal{B}$. Since $x_6$ is not adjacent to $e$ by part 1 above, and $f_6$ is not adjacent to any vertex of $\mathcal{V} \cap \mathcal{B}$ by part 3 above, we must have that $g_6 \in B_2$ and since $g_6$ is adjacent to $f_6 \in E_1$, we have that $g_6 \in E_2$ by Proposition 4(1).

Any $g_7 \in E_2 \cap B_2$ must be adjacent to $e$. $g_7$ must have some neighbour $h_7 \in E_1$ by Proposition 4(1). If $h_7 \in E_1 \cap B_1 = \{e\}$ then we are done. If $h_7 \in E_1 \cap R_1$ then note that $h_7$ is adjacent to $e$ by Lemma 1. Then $g_7, e \in \mathcal{B}$ are both adjacent to $h_7 \in \mathcal{R}$, so $e$ and $g_7$ must be adjacent.

Each $x_8 \in V_1 \cap R_1$ has at most one neighbour in $E_2 \cap B_2$. Suppose for contradiction that $x_8$ were adjacent to two distinct vertices $g_8, h_8 \in E_2 \cap B_2$. Then $e, g_8 \in \mathcal{E}$ must have a common neighbour $y_8 \in \mathcal{V}$ and this common neighbour must be in $V_1 \cap B_1$, since no vertex of $V_2$ can be adjacent to $e \in E_1$ by Proposition 4(2) and we have shown that $e$ is not adjacent to any vertex of $V_1 \cap R_1$. $x_8, y_8 \in \mathcal{V}$ are both adjacent to $g_8 \in \mathcal{E}$, so $x_8$ and $y_8$ must be adjacent. Then $y_8, h_8 \in \mathcal{B}$ are both adjacent to $x_8 \in \mathcal{R}$, but if $y_8$ and $h_8$ are adjacent then we have that $g_8, h_8 \in \mathcal{E}$ are both adjacent to both of $x_8, y_8 \in \mathcal{V}$, contradicting linearity of $(\mathcal{V}, \mathcal{E})$. Note that by part 1 above $x_8$ cannot be adjacent to any vertex of $E_1 \cap B_1$ and part 4 is proved.

Note that although the above lemmas refer to a red vertex $v \in \mathcal{V} \cap \mathcal{R}$, by swapping the names $\mathcal{R}$ and $\mathcal{B}$ and/or $\mathcal{V}$ and $\mathcal{E}$ we can obtain similar results for all vertices of $V(G)$. In the next section we will provide a more general classification of the local behaviour of a recolouring in the case the $(\mathcal{V}, \mathcal{E})$ is linear, which will not depend on fixing $v \in \mathcal{V} \cap \mathcal{R}$.

5. Uniqueness of the total graph for linear hypergraphs

We now greatly extend the results of [3]. We will show that although the total graph of any linear hypergraph $H$ may have many total partitions, each total partition of $T(H)$ uniquely defines $H$, up to isomorphism and duality.

In [3] it is shown that

**Theorem 3** (Behzad and Radjavi [3]). Let $G_1, G_2$ be simple graphs. $T(G_1) \cong T(G_2) \iff G_1 \cong G_2$.

**Proof.** See [3]. \(\square\)
We will extend this result to linear hypergraphs using generalisations of the structural properties from Section 4.

The only two connected simple graphs whose total graphs have more than one total partition are the cycle $C_n$ and the complete graph $K_n$, and in fact Behzad and Radjavi go on to prove that for any connected graph $G$ which is not a cycle or a complete graph, the total partition of $T(G)$ is unique. In this section we will see that although there may be several total partitions of the total graph of a linear hypergraph $H$, all of these partitions induce a hypergraph isomorphic to $H$ or its dual.

Let $G$ be a simple connected graph, which is not a complete graph. Let $(\mathcal{V}, \mathcal{E})$ be a linear total partition of $V(G)$, which induces linear hypergraph $H$. Let $(\mathcal{B}, \mathcal{R})$ be an arbitrary total partition of $V(G)$ which may be nonlinear.

The results of Section 4 were specialised to refer to a vertex $v \in \mathcal{V} \cap \mathcal{R}$ for notational convenience. Note that there were only three possibilities for the neighbourhood of a vertex $v \in \mathcal{V} \cap \mathcal{R}$, given in Lemmas 2, 4 and 5. By swapping $\mathcal{R}$, $\mathcal{B}$ and/or $\mathcal{V}$, $\mathcal{E}$, where necessary, we may arrive at the more general definitions which follow. Throughout this section, we will use two equivalence relations on $V(G)$, $\sim_\mathcal{V}$ and $\sim_\mathcal{E}$, where for $v, w \in V(G)$, $v \sim_\mathcal{V} w \iff v, w \in \mathcal{V}$ or $v, w \in \mathcal{E}$ and $v \sim_\mathcal{E} w \iff v, w \in \mathcal{E}$ or $v, w \in \mathcal{B}$.

**Definition 2.** Let $x \in V(G)$. Then we define $x$ to be

- **type 0** if
  $$N_G(x) \cap \mathcal{V} = N_G(x) \cap \mathcal{B}$$
  or
  $$N_G(x) \cap \mathcal{E} = N_G(x) \cap \mathcal{R},$$

- **type 1** if all neighbours $y$ of $x$ with $x \sim_\mathcal{E} y$ have $x \sim_\mathcal{V} y$.
- **type 2** if there is exactly one neighbour $y$ of $x$ with $x \sim_\mathcal{E} y$ and $x \sim_\mathcal{V} y$.

Note that type 0 vertices correspond to those described in Lemma 2, type 1 vertices correspond to those described in Lemma 4 and type 2 vertices correspond to those in Lemma 5. The lemmas show that all vertices of $G$ must belong to one and only one of the above types.

We can now rephrase the results of Lemmas 2, 4 and 5 more generally.

**Proposition 6.** If there is some $x \in V(G)$ of type 0 then $(\mathcal{V}, \mathcal{E}) = (\mathcal{R}, \mathcal{B})$ or $(\mathcal{V}, \mathcal{E}) = (\mathcal{B}, \mathcal{R})$.

**Proof.** This follows immediately from Theorem 2. \qed

**Proposition 7.** If $x$ is type 1 then:

1. All neighbours $y$ of $x$ with $x \sim_\mathcal{V} y$ and $x \sim_\mathcal{E} y$ are type 1.
2. All neighbours $y$ of $x$ with $x \sim_\mathcal{V} y$ and $x \sim_\mathcal{E} y$ are type 2.
3. All neighbours $y$ of $x$ with $x \cong y$ have $x \sim y$ and are type 2. There must exist at least one such neighbour.

**Proof.** Parts 1–3 follow from the corresponding parts of Lemma 4. The existence of a vertex as described in part 3 follows from Proposition 5, since $x$ must have neighbours in both $\mathcal{V}$ and $\mathcal{E}$. □

**Proposition 8.** If $x$ is type 2 then:

1. There is exactly one neighbour $y$ of $x$ with $x \cong y$ and $x \sim y$, which is type 2.
2. All neighbours $y$ of $x$ with $x \sim y$ and $x \cong y$ are type 1.
3. All neighbours $y$ of $x$ with $x \sim y$ and $x \sim y$ are type 1. There must exist at least one such neighbour.
4. All neighbours $y$ of $x$ with $x \sim y$ and $x \sim y$ are type 2.

**Proof.** Parts 1–4 follow from the corresponding parts of Lemma 5. □

We are now ready to prove an extension to Proposition 6.

**Proposition 9.** If $(\mathcal{V}, \mathcal{E}) \neq (\mathcal{B}, \mathcal{A})$ and $(\mathcal{V}, \mathcal{E}) \neq (\mathcal{B}, \mathcal{A})$ then every vertex of $V(G)$ is either type 1 or type 2. Further, there exist vertices of both types in $V(G)$.

**Proof.** All vertices of $G$ must be type 0, 1 or 2. By Proposition 6 there can be no type 0 vertex if $(\mathcal{V}, \mathcal{E}) \neq (\mathcal{B}, \mathcal{A})$ and $(\mathcal{V}, \mathcal{E}) \neq (\mathcal{B}, \mathcal{A})$. Now by Proposition 7 above, each type 1 vertex must have some type 2 neighbour. By Proposition 8, each type 2 vertex must have some type 1 neighbour. □

Our isomorphism will regard type 1 vertices as being 'static'. Adjacent pairs $v, w$ of type 2 vertices with $v \cong w$ and $v \sim w$ will be swapped. The next proposition shows us that this will give a mapping from $\mathcal{V}$ into $\mathcal{B}$ and from $\mathcal{E}$ into $\mathcal{A}$ and vice versa.

**Proposition 10.** Assume $(\mathcal{V}, \mathcal{E}) \neq (\mathcal{A}, \mathcal{B})$ and $(\mathcal{V}, \mathcal{E}) \neq (\mathcal{B}, \mathcal{A})$. Then there exists some type 1 vertex $x \in V(G)$. By reversing the roles of $(\mathcal{A}, \mathcal{B})$ and/or $(\mathcal{V}, \mathcal{E})$ we may assume that $x \in \mathcal{V} \cap \mathcal{A}$. Then

1. All vertices of $\mathcal{V} \cap \mathcal{A}$ are type 1.
2. All vertices of $\mathcal{E} \cap \mathcal{B}$ are type 1.
3. All vertices of $\mathcal{V} \cap \mathcal{B}$ are type 2.
4. All vertices of $\mathcal{E} \cap \mathcal{A}$ are type 2.

**Proof.** If $(\mathcal{V}, \mathcal{E}) \neq (\mathcal{A}, \mathcal{B})$ and $(\mathcal{V}, \mathcal{E}) \neq (\mathcal{B}, \mathcal{A})$ then by Proposition 9 there is some type 1 vertex $x$, and we may assume without loss of generality that $x \in \mathcal{V} \cap \mathcal{A}$. The proposition is true for $\overline{N}_G(x)$ by Proposition 7. We complete our proof by induction. Suppose the theorem holds for vertices up to distance $k$ from $x$ in $G$, $k \geq 1$. Given
vertex $y$ at distance $k$ from $x$, we will consider the neighbours of $y$ at distance $k + 1$ from $x$, to show that the theorem also holds for these vertices. Consider the following cases:

- $y \in \mathcal{V} \cap \mathcal{R}$: Then by the induction hypothesis $y$ is type 1. If $z \in \mathcal{V} \cap \mathcal{R}$ is adjacent to $y$, then by Proposition 7(1), $z$ is type 1. If $z \in \mathcal{E} \cap \mathcal{R}$ is adjacent to $y$, then by Proposition 7(2), $z$ is type 2. If $z \in \mathcal{E} \cap \mathcal{R}$ is adjacent to $y$, then by Proposition 7(3), $z$ is type 2. If $z \in \mathcal{E} \cap \mathcal{R}$, then it cannot be adjacent to $y \in \mathcal{V} \cap \mathcal{R}$, by Proposition 7(3).

- $y \in \mathcal{V} \cap \mathcal{E}$: Then by the induction hypothesis $y$ is type 1. By Proposition 4(1) applied to both $(\mathcal{V}, \mathcal{E})$ and $(\mathcal{R}, \mathcal{B})$, any $z$ at distance $k + 1$ from $x$ which is adjacent to $y$ must belong to $\mathcal{E} \cap \mathcal{B}$. Then by Proposition 7(1), $z$ must be type 1.

- $y \in \mathcal{E} \cap \mathcal{B}$: Then by the induction hypothesis $y$ is type 2. By Proposition 4(1) applied to $(\mathcal{R}, \mathcal{B})$, any $z$ at distance $k + 1$ from $x$ which is adjacent to $y$ must belong to $\mathcal{E}$. If $z \in \mathcal{V} \cap \mathcal{B}$ is adjacent to $y$, then by Proposition 8(1), $z$ is type 2. If $z \in \mathcal{E} \cap \mathcal{B}$ is adjacent to $y$, then by Proposition 8(3), $z$ is type 1.

Hence any $z$ at distance $k + 1$ from $x$ satisfies the hypothesis and the result is proven by induction. □

We are now ready to prove the proposition leading to our main result.

**Proposition 11.** Assume that there is some vertex in $\mathcal{V} \cap \mathcal{R}$ which is type 1. Then there is an isomorphism $f : V(G) \to V(G)$ such that

$$w, e \in \mathcal{E}, we \in E(G) \iff f(w) \in \mathcal{R}, f(e) \in \mathcal{B}, f(w)f(e) \in E(G).$$

**Proof.** Given that there is some type 1 vertex, by Proposition 9 we have that all vertices are type 1 or type 2, i.e. there are none of type 0.

First we define our isomorphism $f$. If $x \in V(G)$ is of type 1, define $f(x) = x$. If $x \in V(G)$ is type 2 then there is exactly one vertex $y$ adjacent to $x$ such that $x \not\sim y$ and $x \not\not\sim y$. We define $f(x) = y$. $f$ is then a well-defined bijection on $V(G)$. Furthermore, by Proposition 10, $f : \mathcal{V} \to \mathcal{R}$ and $f : \mathcal{E} \to \mathcal{B}$. Note that $f(f(v)) = v$ for all $v \in V(G)$ i.e. $f$ is self-inverse.

Now consider $w, e \in \mathcal{V}, e \in \mathcal{E}$ with $we \in E(G)$. By Proposition 10, $f(w) \in \mathcal{R}, f(e) \in \mathcal{B}$. We will show that $we \in E(G) \iff f(w)f(e) \in E(G)$. Consider the cases

- $w$ and $e$ are both type 1: In this case $f(w) = w, f(e) = e$ and it is clear that $we \in E(G) \iff f(w)f(e) \in E(G)$.

- $w$ and $e$ are both type 2: If $w$ and $e$ receive different colours then $f(w) = e$ and $f(e) = w$ and clearly $f(w)f(e) \in E(G)$. By Proposition 8(3), it is not possible that type 2 $w$ and $e$ both receive the same colour and are adjacent.
• w is type 1, e is type 2: By Proposition 7(3), w and e must have the same colour. Now we know that e is adjacent to exactly one \( x \in V \) which has a different colour from \( e \) and \( w \) and that \( f(e) = x \). w and \( x \) must be adjacent since \( w \in V \) and \( x \in V \) are both adjacent to \( e \in E \). Hence \( f(w)f(e) \in E(G) \).

• w is type 2, e is type 1: By Proposition 8(3), w and e must have the same colour. Now we know that \( w \) is adjacent to exactly one \( e' \in E \) which has a different colour from \( e \) and \( w \) and that \( f(w) = e' \). e and \( e' \) must be adjacent since \( e \in E \) and \( e' \in E \) are both adjacent to \( w \in V \). Hence \( f(w)f(e) \in E(G) \).

For the reverse implication, since \( f \) is self-inverse, we have by the above that \( f(w)f(e) \in E(G) \Rightarrow f(f(w))f(f(e)) \in E(G) \Rightarrow we \in E(G) \).

Let \( I_{\mathcal{V},\mathcal{E}} \) be the bipartite subgraph of \( G \) containing only edges from vertices of \( \mathcal{V} \) to vertices of \( \mathcal{E} \). Let \( I_{\mathcal{E},\mathcal{E}} \) be the bipartite subgraph of \( G \) containing only edges from vertices of \( \mathcal{E} \) to vertices of \( \mathcal{E} \). The proposition above tells us that \( I_{\mathcal{V},\mathcal{E}} \cong I_{\mathcal{E},\mathcal{E}} \). Since the hypergraphs \( H_{\mathcal{V},\mathcal{E}}, H_{\mathcal{E},\mathcal{E}} \) induced by total partitions \((\mathcal{V},\mathcal{E}), (\mathcal{E},\mathcal{E})\), respectively, have isomorphic incidence graphs \( I_{\mathcal{V},\mathcal{E}}, I_{\mathcal{E},\mathcal{E}} \), we must have that \( H_{\mathcal{V},\mathcal{E}} \cong H_{\mathcal{E},\mathcal{E}} \).

We now have sufficient machinery to prove our main result.

**Theorem 4.** Let \( H = (\mathcal{V}, \mathcal{E}) \) be a connected linear hypergraph. Then for arbitrary (possibly nonlinear) hypergraph \( H' \), \( T(H') \cong T(H) \leftrightarrow H' \cong H \) or \( H' \cong H^* \).

**Proof.** If \( H' \cong H \), or \( H' \cong H^* \) then clearly \( T(H') \cong T(H) \). If \( T(H') \cong T(H) = G \) then let \((\mathcal{E},\mathcal{B})\) be the bipartition of \( V(G) \) with the total partition property which induces \( H' \). Of course \((\mathcal{V},\mathcal{E})\) is the bipartition of \( G \) which induces \( H \). If \((\mathcal{V},\mathcal{E}) = (\mathcal{E},\mathcal{B}) \) or \((\mathcal{V},\mathcal{E}) = (\mathcal{B},\mathcal{E}) \) then these two bipartitions induce isomorphic hypergraphs and we are done. Suppose this is not the case. Then by Proposition 9 there must be some type 1 vertex in \( V(G) \). By swapping \( \mathcal{E} \) and \( \mathcal{B} \), and/or swapping \( \mathcal{V} \) and \( \mathcal{E} \) if necessary, we obtain a vertex \( x \in \mathcal{V} \cap \mathcal{E} \) which is type 1 with respect to \((\mathcal{V},\mathcal{E})\). Now Proposition 11 gives us the required isomorphism.

6. Non-unique total partitions

If our total graph \( G \) has multiple connected components then in finding a partition \((\mathcal{E},\mathcal{B})\) of \( V(G) \) with the total partition property, we may exchange \( \mathcal{B} \) and \( \mathcal{E} \) on each connected component and still have a good total partition. If \( G \) has \( k \) connected components then, even if each connected component has a unique total partition, there may be up to \( 2^{k-1} \) nonisomorphic and nondual-isomorphic hypergraphs \( H_1, H_2, \ldots, H_{2^{k-1}} \) induced by total partitions of \( V(G) \), where each component of \( H_i \) will be either isomorphic to, or the dual of the corresponding connected component of \( H_i \).

In [4] it is shown that the only connected simple graphs whose total graphs have a nonunique total partition are the cycle \( C_n \) and the complete graph \( K_n \). Nonuniqueness for \( C_n \) arises since the simple cycles are the only class of graphs which are self-dual,
thus we do not consider that simple cycles have a nonunique partition in our sense. In Fig. 1 we give another example of an infinite family of graphs which have two different linear total partitions. We have written a computer programme to generate such examples and it seems that the phenomenon of having more than one partition of the total graph is not uncommon for small linear hypergraphs.

It is interesting to consider whether Theorem 4 might generalise to all hypergraphs. This is not the case as we show in Figs. 2–4. Hypergraph $H_1$ in Fig. 2 and $H_2$ in Fig. 3 both have a total graph isomorphic to $G_1$ in Fig. 4. Clearly, $H_1 \not\cong H_2$ and $H_1 \not\cong H_2^*$. We know also of other pairs of nonisomorphic hypergraphs with isomorphic total graphs, which do not have loops and which have a more regular structure than $H_1$. 

Fig. 1. Two different (but isomorphic) linear hypergraphs $H, H'$ induced by total partitions of graph $G$. 

---

G

H

H'
7. Algorithms for recognising total graphs

In [9] Gavril considers an algorithm for the problem of deciding whether a given connected simple graph $G$ is the total graph of a simple graph. Using the structural properties proven earlier we can find a more general algorithm to determine whether a graph is the total graph of a linear hypergraph.

Let $G$ be a connected simple graph on $n$ vertices and $m$ edges with maximum degree $\Delta$ and minimum degree $\delta$.

Before we can describe the algorithm we will require several subroutines. First, we will require the subroutine $\text{TOTALEXTEND}(v, R_1, B_1, G)$ given in Section 3, with time complexity $O(m)$. This is essentially the same routine as that used by Gavril. We will also require a routine $\text{ISTOTAL}(\mathcal{R}, \mathcal{B}, G)$ to determine whether partition $(\mathcal{R}, \mathcal{B})$ of $V(G)$ is a total partition. The corresponding routine of Gavril has time complexity
Fig. 3 Hypergraph $H_2$ and its dual which have total graph $G_1$ given in Fig. 4.

Fig. 4. $G_1$, the total graph of $H_1$ in Fig. 2 and $H_2$ in Fig. 3.

$O(m)$ and depends heavily on the properties of the total graph of a simple graph (especially that each vertex of \mathcal{B} has exactly two neighbours in \mathcal{B}). Since we do not have such properties in the general case, we must check that every pair of adjacent vertices of the same colour have a common neighbour of the other colour, with time complexity $O(mA)$ and that if vertex $x$ is adjacent to vertices $y, z$ where $x$ has a different colour from $y, z$ then $y$ and $z$ are adjacent, with time complexity $O(nA^2)$. The corresponding algorithm \textsc{IsLinearTotal}(\mathcal{B}, \mathcal{B}, G), to determine whether $(\mathcal{B}, \mathcal{B})$ is a linear total partition of $V(G)$, has the same time complexity $O(nA^2)$.

Gavril demonstrates an algorithm to find a 'good configuration', that is to find the colouring of the vertices of $\overline{N_G(v)}$ for any given $v$ in time $O(m)$. Again this algorithm depends heavily on the properties of the total graph of a simple graph. For given vertex $v$ we will not be able to find the colouring of $\overline{N_G(v)}$ without trying several such colourings. Note that the number of such colourings is exponential in $\delta$. The algorithm \textsc{FindLinearTotalPartition} which follows will find a linear total partition and will re-
quire O(δ) local colourings to be extended and tested with TOTAL\EXTEND(v, R₁, B₁, G) and ISLINEAR\TOTAL(frared, G). We assume that graph G is not a complete graph. If G is we can test this and output a colouring which colours one vertex red and the rest blue in time O(m).

If G has more than one connected component, we must simply run the algorithm ISLINEAR\TOTAL for each connected component individually. If a total colouring is returned for each connected component then the colourings may be combined to give a total partition.

Algorithm FINDLINEAR\TOTAL\PARTITION(G).

Choose v ∈ V(G) which is of minimum degree δ.

Let Gᵥ be the subgraph of G induced by N(v). Note v /∈ Gᵥ.

Let $\mathcal{A} = \emptyset$.

while $\mathcal{A} \neq V(Gᵥ)$

{ Construct maximal clique $C \in (Gᵥ - \mathcal{A})$.

if ISLINEAR\TOTAL(TOTAL\EXTEND(v, V(Gᵥ) - V(C), V(C), G))

return TOTAL\EXTEND(v, V(Gᵥ) - V(C), V(C), G).

else

for all $y ∈ V(C)$, find a maximal clique $D \in (Gᵥ - \mathcal{A} - (C - \{y\}))$ containing $y$.

if ISLINEAR\TOTAL(TOTAL\EXTEND(V, v(gᵥ) - v(d), v(d), g))

return TOTAL\EXTEND(v, V(Gᵥ) - V(D), V(D), G).

} $\mathcal{A} \leftarrow \mathcal{A} \cup V(C)$.

return ‘No’

FINDLINEAR\TOTAL\PARTITION(G) constructs O(δ) maximal cliques, each requiring a call to TOTAL\EXTEND and ISLINEAR\TOTAL. This gives us a time complexity of O(δ(nA²)). The time complexity of finding v of minimum degree and of finding the maximal cliques is dominated by this term.

**Theorem 5.** On input of a connected simple graph G, which is not a complete graph, the algorithm FINDLINEAR\TOTAL\PARTITION will return a partition of V(G) with the linear total partition property, if such a partition exists.

**Proof.** Assume that there is a partition (R, $\mathcal{B}$) of V(G) with the linear total partition property and without loss of generality assume that vertex v chosen in the first step of algorithm ISLINEAR\TOTAL(G) is coloured red (i.e. belongs to $\mathcal{R}$). Let $R₁$ be the set of vertices of R in Gᵥ and $B₁$ be the set of vertices of $\mathcal{B}$ in Gᵥ. We prove inductively that always $\mathcal{A} \subseteq R₁$. Certainly this is true initially as $\mathcal{A} = \emptyset$. 
We consider the possible forms that the maximal clique $C$ of $G_v$ may take with respect to $R_1$ and $B_1$.

- $C \subseteq B_1$: Since the vertices of $B_1$ induce a clique in $G_v$ we must have $C = B_1$. In this case $\text{TOTALEXtend}(v, V(G_v) - V(C), V(C), G)$ will return a linear total partition and we are done.

- $|C \cap B_1| \leq |C \cap R_1| > 0$: Here we know that $|C \cap B_1| = 1$, since otherwise we would contradict linearity of $(\mathcal{R}, \mathcal{A})$, so let $y$ be the unique vertex of $C \cap B_1$. Since all vertices of $R_1$ adjacent to $y$ induce a clique in $G_v$, we must have that any clique $D$ containing $y$ and no vertex of $C$ or $\mathcal{A}$ must be $B_1$. In this case the inner for loop will find this $y$ and $D$ and $\text{TOTALEXtend}(v, V(G_v) - V(D), V(D), G)$ will return a good total colouring.

- $C \subseteq R_1$: Here we will add $C$ to $\mathcal{A}$ so clearly we maintain $\mathcal{A} \subseteq R_1$.

The algorithm terminates since with each pass through the while loop $|\mathcal{A}|$ increases by at least one. If at any stage we have $\mathcal{A} = V(G_v)$ then we have that in any partition of $V(G)$ with the linear total partition property, $v$ is adjacent to only vertices of the same colour, contradicting Proposition 5. Hence no such colouring exists. $\square$

Note that the algorithm above assumes only that the vertex $v \in V(G)$ has a linear total partition of $\overline{NG}(v)$. Hence if there is any vertex of $G$ which is ‘locally linear’, we can replace the subroutine $\text{ISLINEARTOTAL}$ by $\text{ISTOTAL}$ and by allowing the vertex $v$ in $\text{FINDLINEARTOTALPARTITION}$ to range over the whole of $V(G)$ we can find a total partition of $G$ in time $O(n^2A^3)$.

We can use a similar algorithm to solve in polynomial time other cases where we know that there is some vertex $v$ such that the number of possible partitions of $\overline{NG}(V)$ is polynomially bounded. These include graphs with minimum degree $\delta = O(\log n)$, the total graphs of $r$-regular and the $r$-uniform hypergraphs, where $r = O(\log n)$, and the total graphs of hypergraphs with some hyperedge of rank $O(\log n)$ or some vertex of degree $O(\log n)$.

The general case of determining whether a given simple graph has a total partition remains an interesting open problem. It is unclear whether a problem such as the maximum clique problem could be reduced to it, in which case it would be $NP$-complete, or whether the structural properties of total partitions will allow a polynomial time algorithm.

8. Conclusion and open problems

We have shown that the total graph uniquely defines a linear hypergraph, up to isomorphism and duality. We have given examples to show that if there is no linear total partition of simple graph $G$, then $G$ may be the total graph of two nonisomorphic, nonlinear hypergraphs. It would be interesting to be able to define more exactly for which graphs $G$ this nonuniqueness arises. We intend to study this in a further paper.
We have given a polynomial time algorithm to determine whether $G$ is the total graph of a linear hypergraph. The algorithm has been adapted to determine whether $G$ is the total graph of a wide range of hypergraphs. It remains open, however, whether there is a polynomial time algorithm to determine whether $G$ is the total graph of any hypergraph.

Acknowledgements

I wish to thank Dr. Colin McDiarmid for many helpful discussions and two anonymous referees for their useful suggestions.

References